TANKs.[*]

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[**Preliminary Notes. There might be some notation inconsistency. Treat it with caution! MAKE CORRECTIONS FROM ANDREA**]

Abstract

In this document I review the derivations of the TANK(s) model presented by Bilbiie (2008) and Bilbiie (2019).

[*]I am indebted to Lukas B. Freund who helped me drafting some of the material presented in these notes. All errors and omissions are obviously only my own.

1 Introduction

Throughout, time is discrete and denoted t = 0, 1, 2,...; steady-state variables are without time subscript. Real quantities are in terms of the consumption good and – unless otherwise stated – denoted by lower case letters while nominal variables are denote by capital letters. Log-linear variables in deviation from their steady state will be denoted by a $\wedge$ while log-linear variables in deviations from output steady state by $\sim$.

There is one major difference between the TANK model presented in Bilbiie (2008) and the one in Bilbiie (2019) and it is related to the way that he uses to ensure 0 profits in steady state. This is needed to have a *full insurance* steady state where consumption of both agents is the same. While in 2008 he assumes fix costs in production, in more recent work instead he assumes that the government implements an optimal subsidy inducing marginal cost pricing. This effectively induces a mark-up equal to 1 in steady state and hence 0 profits. I will follow the latter approach here also because it highlights the role of fiscal (re)distribution in these models (which is also one of the findings at the end of the 2008 JET paper.[1]) Here we also assume constant return to scale production as opposed to Bilbiie (2008) who allows for decreasing returns.

2 TANK model

The economy consists of four sectors: households, firms, government and a central bank. The household sector is populated by two different types: **savers** S and **hand-to-mouth** H. The firm sector is standard. The government implements redistributive policies by taxing profits. Nominal rigidities are introduced by assuming that intermediate goods producers face quadratic adjustment costs a la Rotemberg in adjusting their prices. The central bank follows a Taylor type rule to choose the nominal interest rate.

2.1 Households

There is a continuum of households $[0, 1]$, all having the same utility function $U(.)$. There are two types of households: A share λ of households are hand-to-mouth H who work and consume all of their income. The remaining $1 - \lambda$ are savers S who hold bonds and shares in monopolistic firms and get firm profits. Savers price all assets and get all returns, thus there is limited asset market participation.

Savers Savers maximize their lifetime utility subject to their budget constraint, taking prices and wages

[1] In the 2008 he shows that fiscal policy that redistribute dividend income to non-asset holders can restore the standard aggregate demand logic in TANK.

as given:

$$\max_{c_t^S, b_t^S, H_t^S} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{(c_t^S)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \nu^S \frac{(H_t^S)^{1+\varphi}}{1+\varphi} \right) \quad \text{subject to}$$

$$c_t^S + b_t^S = \frac{1-\tau^D}{1-\lambda} d_t + H_t^S w_t + \frac{R_{t-1}}{\Pi_t} b_{t-1}^S,$$

where Π_t is inflation, w_t are real wages, R is the gross nominal interest rate on bonds, d_t are firm profits (with $d_t = (1-\lambda) d_t^S$) and τ^D are taxes levied by the government on dividends. σ is the inter-temporal elasticity of substitution, $\frac{1}{\varphi}$ is the Frish elasticity of labor supply and ν^S indicates how leisure is valued relative to consumption.

The FOCs are:

$$\begin{aligned} {u_t}^{c^S} &= (c_t^S)^{-\frac{1}{\sigma}} \\ u_t^{c^S} &= \beta\, r_t\, E_t u_{t+1}^{c^S} \\ u_t^{H^S} &= -\nu^S (H_t^S)^{\varphi} \\ w_t &= \frac{-u_t^{H^S}}{u_t^{c^S}} \end{aligned}$$

where $r_t = \frac{R_t}{\Pi_{t+1}}$. Rearranging things we have the Euler equation and labor supply for savers:

$$1 = \beta E_t \left[\left(\frac{{c^S}_{t+1}}{{c^S}_t} \right)^{-\frac{1}{\sigma}} r_t \right] \tag{1}$$

$$w_t = \nu^S (H_t^S)^{\varphi} (c_t^S)^{\frac{1}{\sigma}}. \tag{2}$$

Hand-to-mouth Hand-to-mouth households have no assets and thus consume their labor income as well as the transfer they get from the government:

$$\max_{c_t^H, H_t^H} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left(\frac{(c_t^H)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \nu^H \frac{(H_t^H)^{1+\varphi}}{1+\varphi} \right) \quad \text{subject to}$$

$$c_t^H \leq H_t^H w_t + t_t^H.$$

Since preferences are monotonic, households will just consume all of their labor income plus transfers:

$$c_t^H = H_t^H w_t + t_t^H. \tag{3}$$

$$w_t = \nu^H \left(H_t^H\right)^{\varphi} (c_t^H)^{\frac{1}{\sigma}}. \tag{4}$$

2.2 Firms

The firm sector is split into two. There are a representative competitive final goods firm which aggregates intermediate goods according to a CES technology and a continuum of intermediate goods producers that produce different varieties using labor as an input. To the extent to which the intermediate goods are imperfect substitutes, there is a downward-sloping demand for each intermediate variety, giving the intermediate producers some pricing power. However, importantly, intermediate goods producers are subject to some costs in adjusting prices. This generates sticky prices.

Final goods producer Final goods firms maximize profits subject to the production function by taking prices as given. Since final goods firms are all identical, we can focus on one representative firm. These firms bundle the differentiated goods into a final good using a CES technology. The problem of such a representative final goods firm (set up in nominal terms) reads

$$\max_{y_t(i)} P_t y_t - \int_0^1 P_t(i) y_t(i) di \quad \text{s.t.} \quad y_t = \left(\int_0^1 y_t(i)^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}},$$

where ϵ is the elasticity of substitution. When goods are perfectly substitutable $\epsilon \to \infty$, we approach the perfect competition benchmark. Plugging in for the constraint, this rewrites

$$\max_{y_t(i)} P_t \left(\int_0^1 y_t(i)^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(i) y_t(i) di.$$

The first order condition is given by

$$\begin{aligned} \tfrac{\epsilon}{\epsilon-1} P_t \left(\textstyle\int_0^1 y_t(i)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}-1} \tfrac{\epsilon-1}{\epsilon} y_t(i)^{\frac{\epsilon-1}{\epsilon}-1} &= P_t(i) \\ P_t y_t^{\frac{1}{\epsilon}} y_t(i)^{-\frac{1}{\epsilon}} &= P_t(i) \end{aligned}$$

The factor demand of final goods producers is thus given by

$$y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} y_t.$$

From the zero profit condition one can deduce the aggregate price level P_t

$$\begin{aligned} P_t y_t - \int_0^1 P_t(i) y_t(i) di &= 0 \\ P_t y_t - \int_0^1 P_t(i) \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} y_t &= 0 \\ P_t - P_t^{\epsilon} \int_0^1 P_t(i)^{1-\epsilon} &= 0 \\ P_t^{1-\epsilon} &= \int_0^1 P_t(i)^{1-\epsilon} \\ \Rightarrow P_t &= \left(\int_0^1 P_t(i)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}}. \end{aligned}$$

Intermediate goods producer As intermediate goods producers are monopolists, they maximize profits by taking the demand function of final goods firms into account. I consider now the problem of an intermediate goods firm i. For the sake of simplicity the program is split into two sub-problems: the cost minimization and the price setting problem. To find the real cost function, factor costs are minimized subject to the production function. The program of firm i reads

$$\min_{H_t(i)} w_t H_t(i) \quad \text{s.t.} \quad y_t(i) \leq Z_t H_t(i)$$

The first order condition reads

$$w_t = \psi_t(i) Z_t$$

where $\psi_t(i)$ is the corresponding Lagrange multiplier. This multiplier will have the interpretation as real marginal cost – how much will costs change if you are forced to produce an extra unit of output, i.e. $mc_t = \psi_t(i)$. To prove this, let us solve for the Lagrange multiplier as a function of output. We have

$$\psi_t(i) = w_t \frac{1}{Z_t}. \tag{5}$$

By inverting the production function, we have

$$H_t(i) = \frac{y_t(i)}{Z_t}$$

From here, we can derive the cost function

$$C\left(y_t(i), w_t\right) = w_t H_t(i) = w_t \frac{y_t(i)}{Z_t}$$

Thus,

$$C_y\left(y_t(i), w_t\right) = w_t H_t(i) = w_t \frac{1}{Z_t} = \psi_t(i).$$

Also note that because nothing on the LHS depends on i, neither will $\psi_t(i)$. Thus $\psi_t(i) = \psi_t(j) = \psi_t$.

Now that we have found the real cost function, we can move to the intermediate goods firms' price setting problem. Intermediate goods producers set prices to maximize the expected discounted stream of (real) profits (that is real revenue minus real labor input and price adjustment costs). **I also assume that there is an optimal steady-state subsidy in place to induce marginal cost pricing. Essentially, sales are subsidized by τ^S and the subsidy is financed by a lump sum tax levied from the firms.** The price setting problem of the firm thus writes

$$\max_{\{P_{t+k}(i)\}_{k=0}^{\infty}} E_t \sum_{k=0}^{\infty} \mu_{t,t+k} \left(\left(1+\tau^S\right) \frac{P_{t+k}(i)}{P_{t+k}} y_{t+k}(i) - mc_{t+k}(i) y_{t+k}(i) - \frac{\phi_p}{2} \left(\frac{P_{t+k}(i)}{P_{t-1+k}(i)} - 1 \right)^2 y_{t+k} - T_t^f \right)$$
$$\text{s.t.} \left\{ y_{t+k}(i) = \left(\frac{P_{t+k}(i)}{P_{t+k}} \right)^{-\epsilon} y_{t+k} \right\}_{k=0}^{\infty}$$

where $\mu_{t,t+k}$ is the firms stochastic discount factor between period t and $t+k$. This can be rewritten as

$$\Leftrightarrow \max_{\{P_{t+k}(i)\}_{k=0}^{\infty}} E_t \sum_{k=0}^{\infty} \mu_{t,t+k} \left(\left(1+\tau^S\right) \frac{P_{t+k}(i)}{P_{t+k}} \left(\frac{P_{t+k}(i)}{P_{t+k}} \right)^{-\epsilon} y_{t+k} - mc_{t+k}(i) \left(\frac{P_{t+k}(i)}{P_{t+k}} \right)^{-\epsilon} y_{t+k} - \right.$$
$$\left. \frac{\phi_p}{2} \left(\frac{P_{t+k}(i)}{P_{t-1+k}(i)} - 1 \right)^2 y_{t+k} \right)$$

$$\Leftrightarrow \max_{\{P_{t+k}(i)\}_{k=0}^{\infty}} E_t \sum_{k=0}^{\infty} \mu_{t,t+k} \left(\left(1+\tau^S\right) P_{t+k}(i)^{1-\epsilon} P_{t+k}^{\epsilon-1} y_{t+k} - P_{t+k}(i)^{-\epsilon} P_{t+k}^{\epsilon} mc_{t+k}(i) y_{t+k} - \frac{\phi_p}{2} \left(\frac{P_{t+k}(i)}{P_{t-1+k}(i)} - 1 \right)^2 y_{t+k} \right)$$

The FOC reads

$$E_t \left[\mu_{t,t} \left(\left(1+\tau^S\right)(1-\epsilon) P_t(i)^{-\epsilon} P_t^{\epsilon-1} y_t + \epsilon P_t(i)^{-(1+\epsilon)} P_t^{\epsilon} mc_t(i) y_t - \phi_p \left(\frac{P_t(i)}{P_{t-1}(i)} - 1 \right) \frac{y_t}{P_{t-1}(i)} \right) + \mu_{t,t+1} \phi_p \left(\frac{P_{t+1}(i)}{P_t(i)} - 1 \right) \frac{y_{t+1} P_{t+1}(i)}{P_t(i)^2} \right] = 0.$$

Since firms do not have to discount the profits of the current period (i.e. $\mu_{t,t} = 1$) this rewrites

$$\left(1+\tau^S\right)(1-\epsilon) \frac{1}{P_t} \underbrace{\left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} y_t}_{y_t(i)} + \epsilon mc_t(i) \frac{1}{P_t(i)} \underbrace{\left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} y_t}_{y_t(i)} - \frac{1}{P_{t-1}(i)} \phi_p \left(\frac{P_t(i)}{P_{t-1}(i)} - 1 \right) y_t + E_t \left[\mu_{t,t+1} \frac{P_{t+1}(i)}{P_t(i)^2} \phi_p \left(\frac{P_{t+1}(i)}{P_t(i)} - 1 \right) y_{t+1} \right] = 0$$

Because the firms are owned by the savers, the stochastic discount factor is given by $\mu_{t,t+1} = \beta \frac{u_{t+1}{}^{c^S}}{u_t{}^{c^S}} = \beta \left(\frac{c_{t+1}^S}{c_t^S} \right)^{-\frac{1}{\sigma}}$. Using this and multiplying by P_t we arrive at

$$\left(1+\tau^S\right)(1-\epsilon) y_t(i) + \epsilon mc_t(i) \frac{P_t}{P_t(i)} y_t(i) - \frac{P_t}{P_{t-1}(i)} \phi_p \left(\frac{P_t(i)}{P_{t-1}(i)} - 1 \right) y_t + \beta E_t \left[\left(\frac{c_{t+1}^S}{c_t^S} \right)^{-\frac{1}{\sigma}} \frac{P_{t+1}(i) P_t}{P_t(i)^2} \phi_p \left(\frac{P_{t+1}(i)}{P_t(i)} - 1 \right) y_{t+1} \right] = 0.$$

This equation can be interpreted as follows. The first two terms represent the price setting behavior in a standard monopolistic model, i.e. $MR = MC$. The third term can be interpreted as the extra cost at the margin of changing the price today (due to the adjustment costs) and the last term can be viewed as the marginal gain of changing the price today rather than tomorrow (in discounted expected terms).

Thus, firms in the model have a dynamic forward looking view on price setting.

By exploiting the symmetric equilibrium in the above equation (i.e. $y_t(i) = y_t$, $mc_t(i) = mc_t$ and thus $P_t(i) = P_t \, \forall \, t$, which follows from the fact that all firms are identical and face identical demand), one can obtain the following optimal price setting equation

$$\left(1+\tau^S\right)(1-\epsilon)y_t + \epsilon mc_t y_t - \Pi_t \phi_p \left(\Pi_t - 1\right) y_t + \beta E_t \left[\left(\frac{c_{t+1}^S}{c_t^S} \right)^{-\frac{1}{\sigma}} \Pi_{t+1} \phi_p \left(\Pi_{t+1} - 1\right) y_{t+1} \right] = 0$$
$$\Leftrightarrow \left(1+\tau^S\right)(1-\epsilon) + \epsilon mc_t - \Pi_t \phi_p \left(\Pi_t - 1\right) + \beta E_t \left[\left(\frac{c_{t+1}^S}{c_t^S} \right)^{-\frac{1}{\sigma}} \Pi_{t+1} \phi_p \left(\Pi_{t+1} - 1\right) \frac{y_{t+1}}{y_t} \right] = 0$$

From this we can see two sources of inefficiencies. If there are no price adjustment costs, the equation rewrites

$$\epsilon mc_t = \left(1+\tau^S\right)(\epsilon - 1)$$
$$\epsilon MC_t = \left(1+\tau^S\right)(\epsilon - 1) P_t$$
$$\Rightarrow P_t = \underbrace{\frac{\epsilon}{\left(1+\tau^S\right)(\epsilon - 1)}}_{=\mathcal{M}} MC_t$$

Thus, firms charge a constant markup over (nominal MC_t) marginal costs. This distortion can be easily eliminated using a constant subsidy $\tau^S = (\varepsilon - 1)^{-1}$. In this case we have that $\mathcal{M} = 1$ and

$$P_t = MC_t$$

in the limiting case when prices are flexible. However, with sticky prices this markup will be time varying, which introduces another distortion. In steady state, however, there will be no markup, i.e. real marginal costs will be one. By way of summary, optimal firm behavior after having imposed the symmetric equilibrium is characterized by

$$w_t = mc_t \frac{y_t}{H_t}$$
$$\left(1+\tau^S\right)(1-\epsilon) + \epsilon mc_t - \Pi_t \phi_p \left(\Pi_t - 1\right) + \beta E_t \left[\left(\frac{c_{t+1}^S}{c_t^S} \right)^{-\frac{1}{\sigma}} \Pi_{t+1} \phi_p \left(\Pi_{t+1} - 1\right) \frac{y_{t+1}}{y_t} \right] = 0$$
$$y_t = z_t H_t.$$

2.3 Monetary and fiscal policy

The transfer policy is balanced in every period. Thus

$$\lambda t_t^H = \tau^D d_t.$$

As outlined above, the subsidy is financed by a lump-sum tax on the firm. Thus,

$$t_t^F = \tau^S y_t.$$

To close the model, we assume that there is a monetary authority that sets the nominal interest rate according to a simple Taylor rule

$$\frac{R_t}{R} = \left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi} e^{\epsilon_t^m}.$$

2.4 Market clearing

Bond market clearing requires $b_t^S = 0$. Aggregate consumption and aggregate hours are given by

$$c_t = \lambda c_t^H + (1-\lambda) c_t^S$$
$$H_t = \lambda H_t^H + (1-\lambda) H_t^S.$$

Firms profits are given by

$$d_t = \left(1+\tau^S\right) y_t - w_t H_t - \frac{\phi_p}{2}\left(\Pi_t - 1\right)^2 y_t - t_t^f$$
$$= \left(1+\tau^S\right) y_t - mc_t y_t - \frac{\phi_p}{2}\left(\Pi_t - 1\right)^2 y_t - t_t^f$$

Imposing the value for the subsidy, we have

$$d_t = y_t - mc_t y_t - \frac{\phi_p}{2}\left(\Pi_t - 1\right)^2 y_t = \left(1 - mc_t - \frac{\phi_p}{2}\left(\Pi_t - 1\right)^2\right) y_t$$

Using this in the definition of the transfer to H we have

$$t_t^H = \frac{\tau^D}{\lambda}\left(1 - mc_t - \frac{\phi_p}{2}\left(\Pi_t - 1\right)^2\right) y_t.$$

Imposing the market clearing, the households budget constraints read

$$c_t^S = w_t H_t + \frac{1-\tau^D}{1-\lambda} d_t$$
$$c_t^H = w_t H_t + \frac{\tau^D}{\lambda} d_t.$$

Multiplying these by the respective shares and summing, we get

$$\begin{aligned}
\lambda c_t^H + (1-\lambda) c_t^S = \lambda w_t H_t + (1-\lambda) w_t H_t + \tau^D d_t + (1-\tau^D) d_t \\
c_t = w_t H_t + d_t \\
c_t = w_t H_t + y_t - w_t H_t - \frac{\phi_p}{2} \left(\Pi_t - 1\right)^2 y_t \\
c_t = y_t - \frac{\phi_p}{2} \left(\Pi_t - 1\right)^2 y_t
\end{aligned}$$

Thus, the resource constraint reads

$$c_t = y_t - \frac{\phi_p}{2} \left(\Pi_t - 1\right)^2 y_t.$$

2.5 Equilibrium

A general equilibrium for this economy is defined as a sequence of quantities

$\mathcal{Q} = \{y_t, c_t, H_t, H_t^S, H_t^H, mc_t, d_t, c_t^H, c_t^S, t_t^H\}_{t=0}^{\infty}$ and a series of prices $\mathcal{P} = \{w_t, \Pi_t, R_t\}_{t=0}^{\infty}$, and a sequence of shocks $\mathcal{S} = \{Z_t, \epsilon_t\}$ such that

(i) Given a sequence of prices $\mathcal{P}$, and a sequence of shocks $\mathcal{S}$, the sequence of quantities $\mathcal{Q}$ solves the hand-to-mouth, the savers' and the firms' problem.

(ii) Given a sequence of quantities $\mathcal{Q}$, and a sequence of shocks $\mathcal{S}$, the sequence of prices $\mathcal{P}$ clear all the markets.

We have 13 endogenous variables. The equilibrium conditions are given by the following 13 equations.[2]

[2]Note that because of additivity and the fact the utility function elasticities σ and φ are assumed to be the same across households then the aggregate labor supply expression is equivalent to the one of the two households, $w_t = \nu (H_t)^{\varphi} (C_t)^{\sigma}$ where $\nu = \nu^S = \nu^H$.

Equilibrium Conditions		
1:	Labor Supply S	$w_t = \nu^S \left(H_t^S\right)^{\varphi} \left(c_t^S\right)^{\sigma^{-1}}$
2:	Labor Supply H	$w_t = \nu^H \left(H_t^H\right)^{\varphi} \left(c_t^H\right)^{\sigma^{-1}}$
3:	Euler S	$1 = \beta E_t \left[\left(\frac{{c^S}_{t+1}}{{c^S}_t} \right)^{-\frac{1}{\sigma}} \frac{R_t}{\Pi_{t+1}} \right]$
4:	Budget constraint H	$c_t^H = H_t^H w_t + t_t^H$
5:	Transfer H	$t_t^H = \frac{\tau^D}{\lambda} d_t$
6:	Marginal prod. of labor	$w_t = mc_t \frac{y_t}{H_t}$
7:	Phillips Curve	$\left(1+\tau^S\right)(1-\epsilon) + \epsilon mc_t - \Pi_t \phi_p (\Pi_t - 1)$ $+\beta E_t \left[\left(\frac{c_{t+1}^S}{c_t^S} \right)^{-\frac{1}{\sigma}} \Pi_{t+1} \phi_p (\Pi_{t+1} - 1) \frac{y_{t+1}}{y_t} \right] = 0$
8:	Production Function	$y_t = z_t H_t$
9:	Profits	$d_t = \left(1 - mc_t - \frac{\phi_p}{2} (\Pi_t - 1)^2\right) y_t$
10:	Aggregate C	$c_t = \lambda c_t^H + (1-\lambda) c_t^S$
11:	Aggregate H	$H_t = \lambda H_t^H + (1-\lambda) H_t^S$
12:	Resource constraint	$c_t = y_t - \frac{\phi_p}{2} (\Pi_t - 1)^2 y_t$
13:	Taylor Rule	$\frac{R_t}{R} = \left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi} e^{\epsilon_t^m}$

I do not include the S budget constraint because it is implied by Walras law.

2.6 Steady State

We assume that inflation is zero in steady state, $\Pi = 1$, and technology is normalized to unity, $z = 1$. From the Euler equation for bonds of S we have $R = \frac{1}{\beta}$. From the optimal pricing equation, we have $mc = \frac{(1+\tau^S)(\epsilon-1)}{\epsilon} = \mathcal{M}^{-1}$. From the production function, and the resource constraint, we have that $c = y = H$.

For profits, we then have

$$d = (1-\mathcal{M}^{-1})y = (1-\mathcal{M}^{-1})c$$

From the budget constraint for H we have

$$c^H = \mathcal{M}^{-1}c + \frac{\tau^D}{\lambda}d = \mathcal{M}^{-1}c + \frac{\tau^D}{\lambda}\left(1-\mathcal{M}^{-1}\right)c = \left(\mathcal{M}^{-1} + \frac{\tau^D}{\lambda}\left(1-\mathcal{M}^{-1}\right)\right)c$$

From aggregate consumption, we have

$$c^S = \frac{1}{1-\lambda}\left(c - \lambda c^H\right) = \frac{1}{1-\lambda}\left(c - \lambda\mathcal{M}^{-1}c + \tau^D\left(1-\mathcal{M}^{-1}\right)c\right) = \left(\mathcal{M}^{-1} + \frac{1-\tau^D}{1-\lambda}\left(1-\mathcal{M}^{-1}\right)\right)c$$

Thus if there is an optimal subsidy in place, we have $\mathcal{M}^{-1} = 1$ and $c = c^H = c^S$. Under this case of *full steady state insurance* the log-linearized version of the model simplifies substantially.

From the marginal product of labor we have that $w = mc$ and we can use it in the two labor supply equations, together with $c = c^H = c^S = H$, to solve for:

$$\begin{aligned} w &= \nu^H H^{H\varphi} H^{\frac{1}{\sigma}} \\ w &= \nu^S H^{S\varphi} H^{\frac{1}{\sigma}} \\ H &= \lambda H^H + (1-\lambda)H^S. \end{aligned}$$

It is easy to show that because agents have equal consumption and we assume that both agents have the same σ and φ then it must by that household value leisure relative to consumption in the same way (i.e. $\nu^H = \nu^S$) and this means that both households supply the same amount of labor in steady state $H = H^H = H^S = \left(\frac{w}{\nu}\right)^{\frac{\sigma}{1+\varphi\sigma}}$.[3]

[3]However this is not necessarily true outside of the steady state.

2.7 Log-linear Model

We log-linearize the model around the steady state. We assume that inflation is zero in steady state. Variables with a ˆ denote log-deviations from steady state. We log-linearize all variables ($\hat{x}_t = \frac{X_t - X}{X}$) except total profits and transfers, which we linearize and denote as a share of total output, i.e. $\tilde{d}_t = \frac{D_t - D}{Y}$ and $\tilde{t}_t^H = \frac{T_t^H - T^H}{Y}$.

Log-linearized Conditions		
1:	Labor Supply S	$\varphi \hat{H}_t^S = \hat{w}_t - \sigma^{-1}\hat{c}_t^S$
2:	Labor Supply H	$\varphi \hat{H}_t^H = \hat{w}_t - \sigma^{-1}\hat{c}_t^H$
3:	Euler S	$\hat{c}_t^S = E_t \hat{c}_{t+1}^S - \sigma\left(\hat{R}_t - E_t \hat{\Pi}_{t+1}\right)$
4:	Budget constraint H	$\hat{c}_t^H = \hat{H}_t^H + \hat{w}_t + \tilde{t}_t^H$
5:	Transfer H	$\tilde{t}_t^H = \frac{\tau^D}{\lambda}\tilde{d}_t$
6:	Marginal prod. of labor	$\hat{w}_t = \hat{mc}_t + \hat{y}_t - \hat{H}_t$
7:	Phillips Curve	$\hat{\Pi}_t = \beta E_t \hat{\Pi}_{t+1} + \psi \hat{mc}_t$
8:	Production Function	$\hat{y}_t = \hat{z}_t + \hat{H}_t$
9:	Profits	$\tilde{d}_t = -\hat{mc}_t$
10:	Aggregate C	$\hat{c}_t = \lambda \hat{c}_t^H + (1-\lambda)\hat{c}_t^S$
11:	Aggregate H	$\hat{H}_t = \lambda \hat{H}_t^H + (1-\lambda)\hat{H}_t^S$
12:	Resource constraint	$\hat{c}_t = \hat{y}_t$
13:	Taylor Rule	$\hat{R}_t = \phi_\pi \hat{\Pi}_t + \epsilon_t^m$

Table 1: baseline TANK

The slope of the Phillips curve is given by $\psi = \frac{\epsilon}{\phi^p}$. Given additivity and the fact that the two elasticity in utility are the same it is true that the aggregate labor supply looks like the individual ones $\varphi \hat{H}_t = \hat{w}_t - \sigma^{-1}\hat{c}_t$.

Let's simplify the above model by assuming that there are no productivity shocks ($\hat{z}_t = 0$) we have $\hat{y}_t = \hat{c}_t = \hat{H}_t$, $\hat{w}_t = \hat{mc}_t$ and $\tilde{d}_t = -\hat{w}_t$. And from the aggregate labor supply we have

$$\hat{w}_t = (\varphi + \sigma^{-1})\hat{c}_t. \tag{6}$$

Log-linearized Conditions		
1:	Labor Supply S	$\varphi \hat{H}_t^S = \hat{w}_t - \sigma^{-1}\hat{c}_t^S$
2:	Labor Supply H	$\varphi \hat{H}_t^H = \hat{w}_t - \sigma^{-1}\hat{c}_t^H$
3:	Euler S	$\hat{c}_t^S = E_t\hat{c}_{t+1}^S - \sigma\left(\hat{R}_t - E_t\hat{\Pi}_{t+1}\right)$
4:	Budget constraint H	$\hat{c}_t^H = \hat{H}_t^H + \hat{w}_t + \tilde{t}_t^H$
5:	Transfer H	$\tilde{t}_t^H = -\frac{\tau^D}{\lambda}\hat{w}_t$
6:	Phillips Curve	$\hat{\Pi}_t = \beta E_t\hat{\Pi}_{t+1} + \psi\hat{w}_t$
7:	Aggregate C	$\hat{c}_t = \lambda\hat{c}_t^H + (1-\lambda)\hat{c}_t^S$
8:	Aggregate H	$\hat{c}_t = \lambda\hat{H}_t^H + (1-\lambda)\hat{H}_t^S$
9:	Taylor Rule	$\hat{R}_t = \phi_\pi\hat{\Pi}_t + \epsilon_t^m$

Table 2: baseline TANK - no productivity shock

2.8 TANK as in Bilbiie (2019)

Let's derive first the consumption function for H's.

2.8.1 H's consumption function

We start with the following equations for H:

$$\hat{c}_t^H = \hat{w}_t + \hat{H}_t^H + \frac{\tau^D}{\lambda}\tilde{d}_t, \tag{7}$$

$$\hat{H}_t^H = \frac{1}{\varphi}\hat{w}_t - (\varphi\sigma)^{-1}\hat{c}_t^H \tag{8}$$

Combining the H-related equations to write consumption as a function of the wage, taking into account labor supply:

$$\hat{c}_t^H = \frac{\varphi + 1 - \varphi(\tau^D/\lambda)}{\varphi + \sigma^{-1}}\hat{w}_t \tag{9}$$

given the labor supply at aggregate level 6, substituting for the wage in the equation for $\hat{c}_t^H$, and noting that $c_t = y_t$ in a world without government sector, we obtain

$$\hat{c}_t^H = \chi\hat{y}_t, \tag{10}$$

with the elasticity of H's consumption (and income) to aggregate income defined as

$$\chi \equiv \left[1 + \varphi\left(1 - \frac{\tau^D}{\lambda}\right)\right]. \tag{11}$$

Of course, with $\tau^D = 0$, this reduces to $\chi = 1 + \varphi$, which is the version Bilbiie used in previous versions of the NK cross paper.[4]

H's consumption co-moves one-to-one with their income, but not necessarily with aggregate income, and this is the model's keystone: the parameter χ - the elasticity of H's consumption (and income) to aggregate income y_t - which depends on fiscal redistribution and labor market characteristics. From the definition of aggregate consumption, we then have

$$\hat{c}_t^S = \frac{1 - \lambda\chi}{1 - \lambda}\hat{c}_t.$$

From this, one can see that we will be at the RANK benchmark if $\chi = 1$. This is the case e.g. with

[4] For instance: https://www.eui.eu/Documents/DepartmentsCentres/Economics/Seminarsevents/Florin-Bilbiie-paper.pdf

infinitely elastic labor supply $\varphi = 0$, log utility $\sigma = 1$ and no redistribution $\tau^D = 0$.

We can use this result to derive an aggregate Euler equation:

$$\begin{aligned} \frac{1-\lambda\chi}{1-\lambda}\hat{c}_t &= E_t \tfrac{1-\lambda\chi}{1-\lambda}\hat{c}_{t+1} - \sigma\hat{r}_t \\ \hat{c}_t &= E_t\hat{c}_{t+1} - \delta^{-1}\hat{r}_t, \end{aligned} \tag{12}$$

with $\delta = \frac{1-\lambda\chi}{1-\lambda}\frac{1}{\sigma}$.

This illustrates clearly that amplification of MP (understood as changes in $\hat{r}_t = \hat{R}_t - E_t\hat{\Pi}_{t+1}$) occurs with respect to RANK whenever $\chi > 1$.

Moreover we can see how *limited asset market participation* has an effect on the dynamics of the model by affecting the elasticity of aggregate demand to real interest rates δ. And this effect is non-linear as δ change sign whenever λ is higher than the threshold:

$$\lambda^\star = \frac{\tau^D\varphi + 1}{\varphi + 1}. \tag{13}$$

With no redistribution ($\tau^D = 0$) this produce the same picture as in Bilbiie (2008) for $\lambda^\star = \frac{1}{\varphi+1}$ (figure 1). With $\lambda > \lambda^\star$ we are in the IADL region.

Combining 10 with aggregate labor supply 6 and H's labor supply we also get:

$$\begin{aligned} \frac{\hat{c}_t^H}{\chi} &= \frac{\hat{w}_t}{\varphi + \sigma^{-1}} \\ \hat{w}_t = \varphi\hat{H}_t^H &+ \frac{\sigma^{-1}\chi\hat{w}_t}{\varphi + \sigma^{-1}} \end{aligned}$$

therefore we can write:

$$\hat{H}_t^H = \eta\hat{w}_t \tag{14}$$

$$\eta = \frac{\varphi + \sigma^{-1}(1-\chi)}{\varphi + \sigma^{-1}}\frac{1}{\varphi} \tag{15}$$

with η being the elasticity of H's hours to real wage. Again is useful to look at the no redistribution case with $\chi = 1 + \varphi$ so that

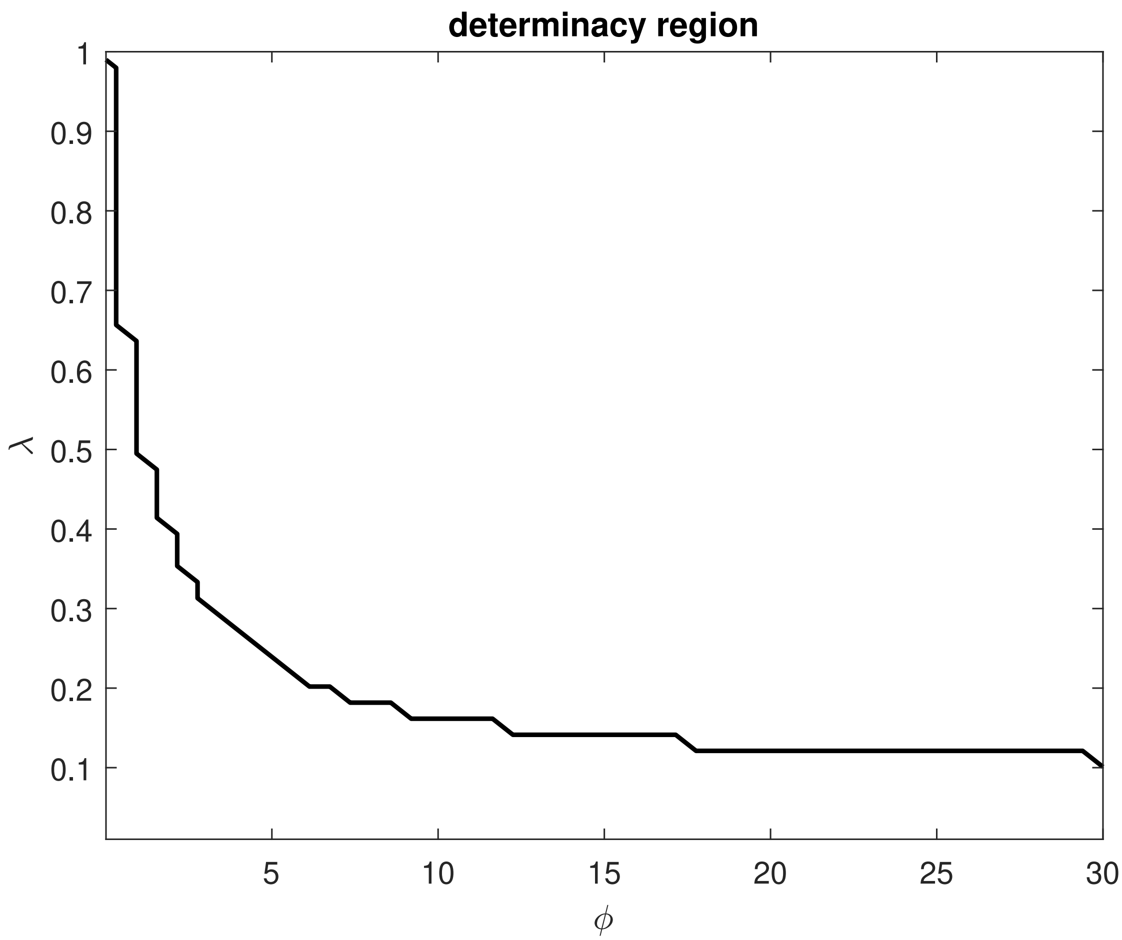

Figure 1: Determinacy region as in Bilbiie (2008)

$$\eta = \frac{1-\sigma^{-1}}{\varphi+\sigma^{-1}}, \tag{16}$$

which shows that with no redistribution ($\tau^D = 0$) and $\sigma = 1$ we get $\hat{H}_t^H = 0$ so hours of the Hand to Mouth are constant.

References

Bilbiie, F. (2008). Limited asset markets participation, monetary policy and (inverted) aggregate demand logic. *Journal of Economic Theory*, **140**(1), 162–196.

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