CIMS Summer School

Advanced Macro-Modeling

Central Bank Communication, Imperfect Credibility, and Optimal Monetary Policy Applications

UNIVERSITY OF SURREY

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Organization of the day

9.30 — 11.00:	Session 1
11.00 — 11.30:	Coffee and Tea
11.30 — 13.00:	Session 2
13.00 — 14.15:	Lunch
14.15 — 15.45:	Session 3
15.45 — 16.15:	Coffee and Tea
16.15 — 17.45:	Session 4

Goals

- Understand and apply concepts of optimal policy and imperfect credibility.
  - Mainly monetary policy but tools can be applied to other frameworks, e.g. fiscal, macro-prudential.
- Provide you with a toolbox that is easy to use and allows you to apply the concepts learned here either in your research or policy work.
- Both theory and computer exercises.

Outline

- Simplest New Keynesian model:
  - Time-inconsistent solution
  - Time-consistent solution
  - This will be derived and explained in a manner consistent with recursive contracts theory and will set the stage for the imperfect commitment settings.
- Central bank communication:
  - State-contingent nature of commitment
  - Targeting rules
- Monetary policy design
  - Benefits of price level targeting
  - Alternative policies
- Imperfect credibility

Outline

- Some theory behind solutions
- Toolkit of imperfect credibility
- Application to large scale models
  - Smets and Wouters AER 2007 model
    - What are the gains of achieving more credibility?
    - How does the possibility of future re-optimizations affect current outcomes and promises?
    - How does imperfect credibility affect the shock propagation, volatilities, and cross-correlations between relevant variables?
    - Does the policy response to some shocks require more commitment? At what stages?

Introduction

- Managing expectations is crucial for determining optimal policy:
  - anchoring inflation expectations,
  - providing forward guidance,
  - speeches, announcements, press releases.
- Time-inconsistency problem (Kydland and Prescott (1977), Barro and Gordon (1983)).

Introduction

- What is the source of the time-inconsistency problem in the simplest New Keynesian model?

$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1} + u_t$$
$$y_t = E_t y_{t+1} - \sigma \left(i_t - E_t \pi_{t+1} \right) + g_t$$

Introduction

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- By managing $E_t \pi_{t+1}$ and $E_t y_{t+1}$, the central bank can influence π_t and y_t.
- In a rational expectations equilibrium, $E_t \pi_{t+1}$ and $E_t y_{t+1}$ need to correspond with actual outcomes in period $t+1$.
- There is an incentive to commit to a policy for $t+1$ just because doing so allows for better outcomes in period t.
- In $t+1$ there is an ex-post incentive to renege.

Commitment and stabilization policy:

$$V(u_t, g_t) = \max_{\{y_t, \pi_t, i_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \{-(\pi_t^2 + \lambda y_t{}^2)\}$$

$$\begin{aligned}
\text{s.t. } \pi_t &= \kappa y_t + \beta E_t \pi_{t+1} + u_t \\
y_t &= E_t y_{t+1} - \sigma \left(i_t - E_t \pi_{t+1}\right) + g_t \\
u_t &= \rho_u u_{t-1} + \varepsilon_{u,t} \\
g_t &= \rho_g g_{t-1} + \varepsilon_{g,t},
\end{aligned}$$

- Output gap target is zero: $\pi_t^2 + \lambda \left(y_t - \bar{y}\right)^2$ with $\bar{y} = 0$
- We don't need the IS equation. Exercise: Can you show this formally?

Commitment and stabilization policy:

Write the Lagrangean:

$$V(u_t, g_t) = \max_{\{y_t, \pi_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \{-(\pi_t^2 + \lambda {y_t}^2) + \gamma_t (\pi_t - \kappa y_t - \beta E_t \pi_{t+1} - u_t)\}$$

Crucial step: At time zero, the central bank decides a plan once and for all.

- The expectation term $E_t \pi_{t+1}$ can, therefore, be decided upon directly.
- Law of iterated expectations: $E_0 E_t \pi_{t+1} = E_0 \pi_{t+1}$

And the problem becomes:

$$V(u_t, g_t) = \max_{\{y_t, \pi_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \{-(\pi_t^2 + \lambda {y_t}^2) + \gamma_t (\pi_t - \kappa y_t - \beta \pi_{t+1} - u_t)\}$$

We can take first order conditions here directly, but we will first rearrange the lagrangean.

Commitment and stabilization policy:

Rearranging parts of constraints:

$$
\begin{aligned}
V(u_t, g_t) = \quad & \gamma_0 \left(\pi_0 - \kappa y_0 - \beta \pi_1 - u_0\right) \\
& + \beta \gamma_1 \left(\pi_1 - \kappa y_1 - \beta \pi_2 - u_1\right) \\
& + \beta^2 \gamma_2 \left(\pi_2 - \kappa y_2 - \beta \pi_3 - u_2\right) + \dots \\
= \quad & [\gamma_0 \left(\pi_0 - \kappa y_0 - u_0\right)] \\
& + \beta[\gamma_1 \left(\pi_1 - \kappa y_1 - u_1\right) - \gamma_0 \pi_1] \\
& + \beta^2[\gamma_2 \left(\pi_2 - \kappa y_2 - u_2\right) - \gamma_1 \pi_2] + \dots
\end{aligned}
$$

1. We can always shift forward the term π_{t+1} because there are infinitely many terms going forward.
   [Note: this would be different in a finite horizon.]
2. The period $t = 0$ is different, because there are no expectations from period $t = -1$ to be shifted forward into $t = 0$.

Commitment and stabilization policy:

Rearranging parts of constraints:

$$\begin{aligned} V(u_t, g_t) = & \quad \gamma_0 (\pi_0 - \kappa y_0 - \beta \pi_1 - u_0) \\ & + \beta \gamma_1 (\pi_1 - \kappa y_1 - \beta \pi_2 - u_1) \\ & + \beta^2 \gamma_2 (\pi_2 - \kappa y_2 - \beta \pi_3 - u_2) + \ldots \\ = & \quad [\gamma_0 (\pi_0 - \kappa y_0 - u_0)] \\ & + \beta [\gamma_1 (\pi_1 - \kappa y_1 - u_1) - \gamma_0 \pi_1] \\ & + \beta^2 [\gamma_2 (\pi_2 - \kappa y_2 - u_2) - \gamma_1 \pi_2] + \ldots \end{aligned}$$

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Commitment and stabilization policy:

$$V(u_t, g_t) = \max_{\{y_t, \pi_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \{ -(\pi_t^2 + \lambda {y_t}^2)$$

$$+ \gamma_t (\pi_t - \kappa y_t - u_t) - \gamma_{t-1} \pi_t \}$$

$$\text{AND } \gamma_{-1} = 0$$

Taking first order conditions:

Commitment and stabilization policy:

$$V(u_t, g_t) = \max_{\{y_t, \pi_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \{ -(\pi_t^2 + \lambda {y_t}^2)$$

$$+ \gamma_t (\pi_t - \kappa y_t - u_t) - \gamma_{t-1} \pi_t \}$$

$$\text{AND } \gamma_{-1} = 0$$

Taking first order conditions:

$$\gamma_t : \pi_t - \kappa y_t - \beta E_t \pi_{t+1} - u_t = 0$$

$$\pi_t : -2\pi_t + \gamma_t - \gamma_{t-1} = 0$$

$$y_t : -2\lambda y_t - \kappa \gamma_t = 0$$

$$\gamma_{-1} = 0$$

Commitment and stabilization policy:

Rearranging the FOCs:

$$-2\pi_t + \gamma_t = 0, \; t = 0$$
$$-2\pi_t + \gamma_t - \gamma_{t-1} = 0, \; t \geq 1$$

- Exactly because the FOCs change from $t = 0$ to $t \geq 1$, there is a time-inconsistency problem. Why?
  - If the FOCs were equal regardless of t then the solution would be time-consistent.
- In period t=1, the central bank would like to implement the FOC of period $t = 0$. Why?
  - Remember, that past lagrange multipliers are associated with past constraints and were just "carried forward" because of expectations.
  - How would we ignore a constraint? We put the lagrange multiplier to zero.

Commitment and stabilization policy:

A side note for later:

We can express the past lagrange multiplier as a function of other endogenous variables:

$$\gamma_t = -2\frac{\lambda}{\kappa}y_t \text{ for } t \geq 0.$$

But again note that:

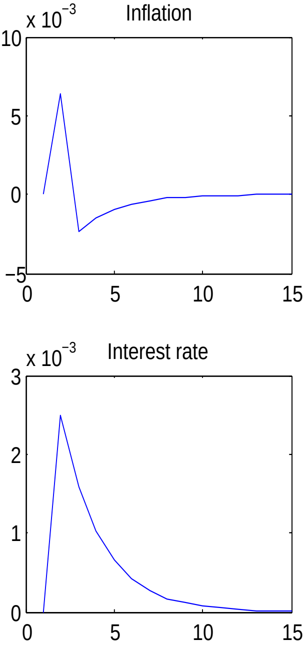
$$\gamma_{-1} = 0 \neq -2\frac{\lambda}{\kappa}y_{-1}.$$

The time-inconsistency is still there.

Commitment and stabilization policy:

- The system had no endogenous persistence. But commitment introduces persistence through γ_{t-1}.
- Past lagrange multipliers γ_{t-1} summarize the shadow value of past promises that need to be fulfilled today.
- The lagged lagrange multiplier γ_{t-1} is not a PHYSICAL state variable, it measures past promises.
- The central bank would like to reset this variable to 0 (welfare of unconstrained maximization is higher than constrained one).

IRF i.i.d. cost push shock under commitment



IRF i.i.d. cost push shock under commitment

Impulse response function to a i.i.d. cost push shock – there is no endogenous or exogenous persistence in the model.

- Previous figure shows persistence....
- There is a part in the graph where we can see that the central bank is fulfilling previous promises. If the central bank could, it would like to reoptimize. When (which period)?

IRF i.i.d. cost push shock under commitment

- Above we discussed why it is not optimal ex-post. Can you describe why such promises are optimal from an ex-ante perspective?

$$V(u_t, g_t) = \max_{\{y_t, \pi_t, i_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \{-(\pi_t^2 + \lambda {y_t}^2)\}$$

$$\text{s.t. } \pi_t = \kappa y_t + \beta E_t \pi_{t+1} + u_t$$

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- A planner equates/distributes the trade-offs of shocks or distortions across different 1) variables 2) states of nature (insurance).

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$$\text{s.t. } \pi_t = \kappa y_t + \beta E_t \pi_{t+1} + u_t$$

- A planner equates/distributes the trade-offs of shocks or distortions across different 1) variables 2) states of nature (insurance).
- A planner with commitment also equates/distributes the trade-offs across time.

IRF i.i.d. cost push shock under commitment

Simple illustrative example for intuition:

- Intuitively, think that due to the cost-push shock, inflation today goes up today by 2 units.
- Since the cost push shock is i.i.d., tomorrow inflation can be at target.
- For simplicity, we set $\beta = 1$ and focus on inflation only.

$$\overbrace{(2)^2}^{\text{Costs today}} + \overbrace{(0)^2}^{\text{Costs tomorrow}} = \overbrace{4}^{\text{Costs total}}$$

The central bank under commitment improves on this by promising to be tough tomorrow on inflation. This implies a reduction in inflation today.

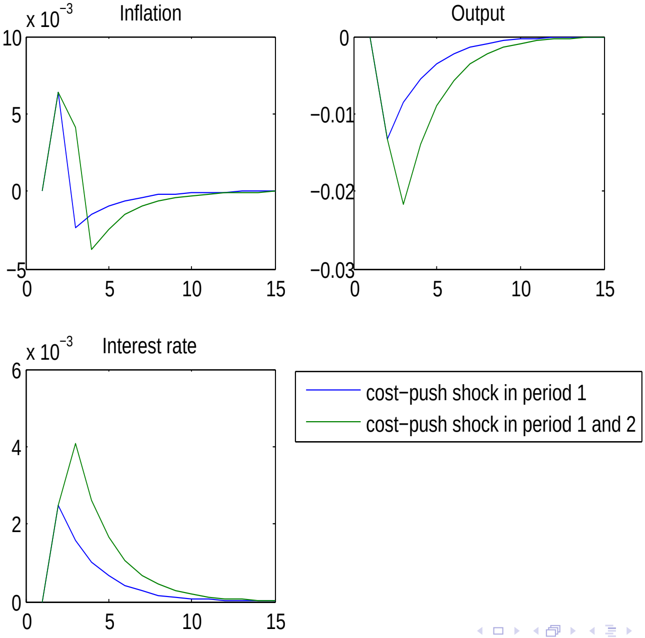
$$\underbrace{\pi_t}_{\searrow} = \kappa y_t + \beta \underbrace{E_t \pi_{t+1}}_{\searrow} + u_t$$

$$\overbrace{(1.5)^2}^{\text{Costs today}} + \overbrace{(-0.5)^2}^{\text{Costs tomorrow}} = \overbrace{2.5}^{\text{Costs total}}$$

Time-inconsistency and state-contingency

- We often associate commitment with lack of flexibility.
- We commit into something tomorrow that we will do regardless of what happens tomorrow.
- Hence the expression "Commitment vs discretion".
  - Commitment allows you to influence expectations.
  - But under commitment you can not react to future unforeseen contingencies
- This is not what we are doing here.
- We are doing commitment that is state-contingent:
  - Commitment allows the planner to decide today regarding tomorrow;s policy, but what is being decided and what is implemented does depend on shocks.
  - Planner commits to implement policy action A if shock is x, and implement policy action B if shock if y.

Time-inconsistency and state-contingency



Time-inconsistency and state-contingency

Blue line

- The shocks are $u_1 = \sigma_u, u_2 = 0, u_3 = 0, u_4 = 0$...
- At period $t = 1$, the planner knows the shock $u_1 = \sigma_u$ and handles it with a commitment policy.
- $u_2 = 0, u_3 = 0$... is the expected path, so this one is easy to plot to see the effects of promises.
- However, the commitment policy designed at $t = 1$ implies a different π_{t+2} for each level of the cost push shock.

Time-inconsistency and state-contingency

Green line:

- The shocks are $u_1 = \sigma_u, u_2 = \sigma_u, u_3 = 0, u_4 = 0...$
- Blue and green lines are not equal.
  - This makes the point that the promises are state-contingent.
- At $t = 2$ there is a similar type of response to that at $t = 1$.

Promises are state-contingent. However, unexpected shocks do not wipe out previous promises made at $t = 1$.

- Where can you see that?
  - Note: compare inflation level at $t = 2$ green line with $t = 1$ blue line.
- Hence, we can describe the promise at $t = 1$ as: inflation will respond to economic developments but inflation will be set at a lower level than usual.

Time-inconsistency and state-contingency

Where was it in the math that commitment was state contingent?

Time-inconsistency and state-contingency

Where was it in the math that commitment was state contingent?

- Answer: When we applied the law of iterated expectations and carried forward the lagrange multipliers.
- If we do those steps “manually”, you can see that we have one promise for each shock.

Time-inconsistency and state-contingency

$$= \max_{\{y_t, \pi_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \{ -(\pi_t^2 + \lambda {y_t}^2) + \gamma_t \left(\pi_t - \kappa y_t - \beta E_t \pi_{t+1} - u_t \right) \}$$

Write probabilities explicitly:

$$= \max_{\{y_t, \pi_t\}} \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \beta^t \{ -P\left(\omega^t\right) (\pi_t^2 + \lambda {y_t}^2) + P\left(\omega^t\right) \gamma_t \left(\pi_t - \kappa y_t - \beta \sum_{\omega^{t+1} \in \Omega^{t+1}} P\left(\omega^{t+1} | \omega^t\right) \pi_{t+1} - u_t \right) \}$$

Put terms (t) apart from terms (t+1):

$$= \max_{\{y_t, \pi_t\}} \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \beta^t \{ -P\left(\omega^t\right) (\pi_t^2 + \lambda {y_t}^2) + P\left(\omega^t\right) \gamma_t (\pi_t - \kappa y_t - u_t) - P\left(\omega^t\right) \gamma_t \beta \sum_{\omega^{t+1} \in \Omega^{t+1}} P\left(\omega^{t+1} | \omega^t\right) \pi_{t+1} \}$$

Arrange conditional expectations part:

$$= \max_{\{y_t, \pi_t\}} \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \beta^t \{ -P\left(\omega^t\right) (\pi_t^2 + \lambda {y_t}^2) + P\left(\omega^t\right) \gamma_t (\pi_t - \kappa y_t - u_t) \underbrace{- \gamma_t \beta \sum_{\omega^{t+1} \in \Omega^{t+1}} P\left(\omega^{t+1}\right) \pi_{t+1}} \}$$

Finally shift terms forward:

$$= \max_{\{y_t, \pi_t\}} \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega^t} \beta^t \{ -P\left(\omega^t\right) (\pi_t^2 + \lambda {y_t}^2) + P\left(\omega^t\right) \gamma_t (\pi_t - \kappa y_t - u_t) - \boldsymbol{\gamma_{t-1} P\left(\omega^t\right) \pi_t} \}$$

Time-inconsistency and state-contingency

- The previous slide also shows more clearly why it is appropriate to put forward the lagrange multipliers.
  - At each node in time t we have enough terms for $t + 1$.
- Consider that we have one shock with two realizations.
  - In each node, the “tree of events” unfolds into two sub-branches.
  - In each node, the expectations term has two elements that can go into those two sub-branches.

Time-consistent policy:

Two changes:

1. The central bank can only decide on policies for $t = 0$.
   - Policies at $t = 1$ are decided by another entity: different selves or a different CEO of the central bank.
2. The central bank cannot affect private expectations directly.

$$V^D(u_t, g_t) = \max_{\{y_t, \pi_t\}} E_0 \left\{ -(\pi_t^2 + \lambda {y_t}^2) + \beta E_t V^D(u_{t+1}, g_{t+1}) \right\}$$

$$\text{s.t. } \pi_t = \kappa y_t + \beta E_t \Psi\left(u_{t+1}, g_{t+1}\right) + u_t$$

Time-consistent policy:

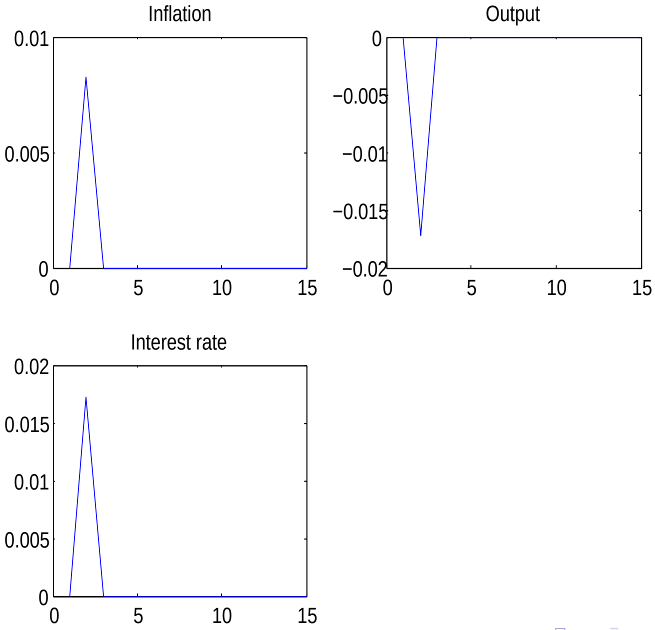
Taking FOCs:

$$\gamma_t : \pi_t - \kappa y_t - \beta E_t \pi_{t+1} - u_t = 0$$
$$\pi_t : -2\pi_t + \gamma_t = 0$$
$$y_t : -2\lambda y_t - \kappa \gamma_t = 0$$

Note the following:

- FOCs are the same for $t = 0$ and $t \geq 1$. Policy is time consistent.
- Previous period lagrange multipliers are gone.
- There is no endogenous persistence.

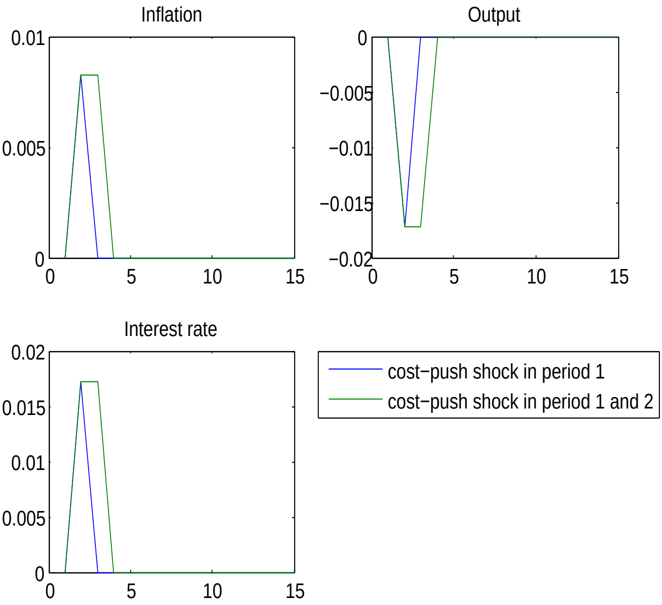
IRF i.i.d. cost push shock under discretion



IRF i.i.d. cost push shock under discretion

- Since shock is not persistence, effects on variables disappear immediately.

IRF i.i.d. cost push shock under discretion

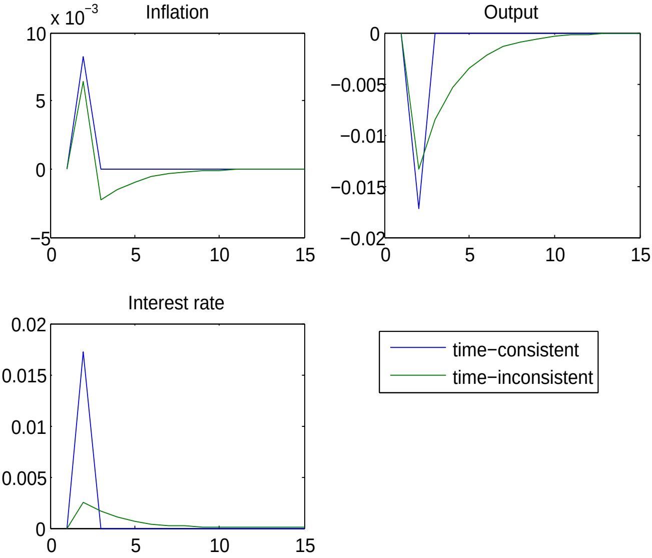


IRF i.i.d. cost push shock under discretion

- Note that now green line in second period is at the same height of blue line in first period.

What about the direct comparison of commitment and discretion?

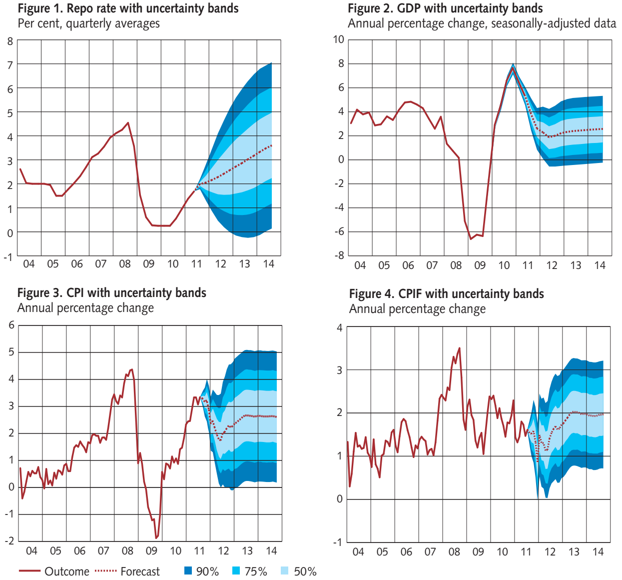
IRF i.i.d. cost push shock under commitment and discretion Go



Commitment and communication strategies

- Discretion policy does not require communication: in future periods, the central bank will do what it always does.
- Commitment requires communication:
  - The benefits of commitment are only present because the central bank communicates future policy actions to the public and thereby manages effectively private sector expectations.
    - Important: Keeping promises that were not communicated is a bad idea.
  - Commitments are state-contingent and depend on the evolution of the economy.
- It's easy in the model but, in practice, it does not seem easy to communicate commitment...

Riksbank's Fan charts



Note. The uncertainty bands in the figures are based on historical forecast errors, see the article "Calculation method for uncertainty bands" in MPR 2007:1.

Sources: Statistics Sweden and the Riksbank

Commitment and communication strategies

Is it easy to distinguish commitments from forecasts?

- Think that the central bank also wants to communicate the likely course of the economy...Why?
  - One of the reasons that we believe in Rational Expectations (or believe in it somewhat) is that certain institutions can compute forecasts for us.

Commitment versus communicating the state of the economy may be hard to distinguish

- This became a big issue during the Great Recession and because of the ZLB.
- Two papers that try to address this:
  - Bodenstein, M. , James Hebden, and Ricardo Nunes "Imperfect Credibility and the Zero Lower Bound on the Nominal Interest Rate", Journal of Monetary Economics, 2012.
  - Campbell, J., Jonas Fisher, Alejandro Justiniano, and Leonardo Melosi"Forward guidance and macroeconomic outcomes since the financial crisis," in NBER Macroeconomics Annual 2016.

Additional Issues of Implementing Optimal Policy

- Communication and State-contingency ✓

Next:

- Implementation of optimal policy. Interest rate rules.
- Targeting Rules.
- Expectational interest rate rules.

Additional Issues of Implementing Optimal Policy

- Once we solve the optimal policy outcomes. We can obtain laws of motion for inflation and output as a function of the state variables:

$$\pi_t = ay_{t-1} + bu_t \tag{1}$$
$$y_t = cy_{t-1} + du_t. \tag{2}$$

- One can then ask the question: How do we implement this outcome?

Additional Issues of Implementing Optimal Policy

- We often think about interest rate rules.
- We can plug the laws of motion (1) and (2) into an interest rate rule of the type:

$$i_t = \phi_\pi \pi_t + \phi_y y_t.$$

- In these type of exercises, we often find that the equilibrium is not unique.
- There are several versions of this result...
  - One can also find coefficients ϕ_π and ϕ_y that are consistent with the optimal policy.
  - But again some of these interest rate rules do not lead to an unique equilibrium.
- We now discuss an alternative: Targeting Rules.

Targeting rules

Targeting rules are the FOCs of the system written in a compact way

$$\pi_t = -\frac{\lambda}{\kappa}(y_t - y_{t-1})$$

This seems very easy, as simple as an interest rate rule such as $i_t = \phi_\pi \pi_t + \phi_y y_t$.

When we consider the system with an interest rate rule we do:

$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1} + u_t$$
$$y_t = E_t y_{t+1} - \sigma(i_t - E_t \pi_{t+1}) + g_t$$
$$i_t = \phi_\pi \pi_t + \phi_y y_t$$

Now we do:

$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1} + u_t$$
$$y_t = E_t y_{t+1} - \sigma(i_t - E_t \pi_{t+1}) + g_t$$
$$\pi_t = -\frac{\lambda}{\kappa}(y_t - y_{t-1})$$

Targeting rules

- Of course this system implements optimal policy. It is the system of equations for optimal policy!
- The economics of this are the following: The central bank announces that inflation will be kept at a level inversely proportional to the growth rate of the output-gap.
- If the public believes and understands the system of equations then this is all correct!

Targeting rules

The targeting rule purposefully abstracts from the implementation.

- Advantages
  - Several instruments (interest rates, communication, forward guidance, quantitative easing, money)
  - Are you a micro-manager?
  - Do you tell the taxi-driver the final destination or how many times to turn left and right, and which pedals to press?
  - Isn't this what the public should actually know?
- Disadvantages
  - Sure... Ok... but how do we implement it?
  - If you are the client in the taxi or in the restaurant, you care about the targeting rule. But... What if you are the driver or the chef?

Targeting rules

Targeting rules are certainly useful, but some considerations:

- The model consists of 2 constraints. The target rule is "just" 1 equation – nearly as complicated as the entire description of private agents behavior.
- If one considers more complex models, targeting rules start looking very complicated.
- Sometimes it may not be immediate to eliminate/substitute lagrange multipliers.

Targeting rules and Expectational Interest Rate Rules

- What is the relation between Targeting Rules and Instrument Rules?
- There is not necessarily a strict relation if the interest rate rule is too simple. Remember that sometimes it is not possible to obtain a unique equilibrium with some interest rate rules.
- Evans and Honkapohja, 2003. Review of Economic Studies.and Evans and Honkapohja (2006). Scandinavian Journal of Economics make an important point.

Targeting rules and Expectational Interest Rate Rules

Reverse engineer a rule such that the system:

$$y_t = E_t y_{t+1} - \sigma\left(i_t - E_t \pi_{t+1}\right) + g_t$$
$$i_t = F(...)$$

is equivalent to the system:

$$y_t = E_t y_{t+1} - \sigma\left(i_t - E_t \pi_{t+1}\right) + g_t$$
$$\pi_t = -\frac{\lambda}{\kappa}\left(y_t - y_{t-1}\right).$$

This is always possible to do. Rearrange the IS equation:

$$i_t = E_t \pi_{t+1} - \frac{1}{\sigma}\left(y_t - E_t y_{t+1} - g_t\right).$$

Now sum the FOC: $\pi_t + \frac{\lambda}{\kappa}\left(y_t - y_{t-1}\right) = 0$

$$i_t = E_t \pi_{t+1} - \frac{1}{\sigma}\left(y_t - E_t y_{t+1} - g_t\right) + \pi_t + \frac{\lambda}{\kappa}\left(y_t - y_{t-1}\right).$$

Designing Monetary Policy objectives:

- Assumption 1: It is not possible to conduct time-inconsistent policy, the central bank would renege.
- Assumption 2: Society can design a central bank at the beginning of time, and will abstain from interfering with monetary policy.

Are these two assumptions compatible?

- McCallum, B. T., 1995 AER critique
- Any defense? Go

Designing Monetary Policy objectives:

Society's utility:

$$-E_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \lambda {y_t}^2)$$

Central bank's utility:

$$-E_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \lambda_{CB} {y_t}^2)$$

λ_{CB} is chosen by society to maximize its own utility. (theory of the second best)

Appointing a conservative central bank

The FOC of time-inconsistent policy is:

$$\pi_t = -\frac{\lambda}{\kappa}\left(y_t - y_{t-1}\right).$$

The FOC of the central bank is:

$$\pi_t = -\frac{\lambda_{CB}}{\kappa} y_t.$$

- This policy design does not create persistence and is not very helpful as a shock stabilization policy.
- This policy is more helpful to mitigate the inflation bias due to a positive output-gap target $\bar{y} > 0$.
  - Time inconsistent FOC: $\pi_t = -\frac{\lambda}{\kappa}\left(y_t - y_{t-1}\right)$
  - Central Bank FOC: $\pi_t = -\frac{\lambda_{CB}}{\kappa}(y_t - \bar{y})$
    - Lower λ_{CB} makes "wrong" term $[\lambda_{CB}\bar{y}]$ become smaller.

Price-level targeting:

Intuition, a FOC of the type

$$p_t = -\frac{\lambda}{\kappa} y_t$$

implies the correct law of motion:

$$\pi_t = -\frac{\lambda}{\kappa} \left(y_t - y_{t-1} \right).$$

The central bank preferences

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ (p_t)^2 + \lambda_{CB} {y_t}^2 \right\}$$

Price-level targeting:

Intuition in relation to the i.i.d. cost push shock: Commitment Dynamics

- Step 1: When the cost-push shock hits, inflation goes up and the price level goes above target.
- Step 2: The central bank immediately faces an incentive to bring the price level down next period. Therefore, inflation next period will be below target mimicking the commitment solution.

Speed Limit Policies

The FOC of time-inconsistent policy is

$$\pi_t = -\frac{\lambda}{\kappa}(y_t - y_{t-1})$$

The FOC of time-consistent policy is

$$\pi_t = -\frac{\lambda}{\kappa} y_t$$

What if we substitute in the objective function y_t by $(y_t - y_{t-1})$:

$$E_0 \sum_{t=0}^{\infty} \beta^t \{-(\pi_t^2 + \lambda_{CB}(y_t - y_{t-1})^2)$$

Speed Limit Policies

Intuition in relation to the i.i.d. cost push shock: Commitment Dynamics

- Step 1: When cost-push shock hits, inflation goes up and a recession is in place.
- Step 2: The central bank objective IS NOT to end the recession immediately ($y_{t+1} = 0$). The objective is to smooth the recovery $(y_{t+1} - y_t)$.
- Step 3: In doing so, the recovery is slower and inflation is below target.

Other Policies

Looking at the objectives, FOCs, and IRFs what other types of delegated objectives do you think would have a chance at improving welfare?

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Looking at the objectives, FOCs, and IRFs what other types of delegated objectives do you think would have a chance at improving welfare?

- What about interest rate inertia through a term $(i_t - i_{t-1})^2$?
- What about nominal income growth targeting through a term $[(p_t + x_t) - (p_{t-1} + x_{t-1})]^2$?
  - Note that: $(p_t + x_t) - (p_{t-1} + x_{t-1}) = \pi_t + x_t - x_{t-1}$

Other Policies

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This debate is still ongoing...

- For a recent proposal of price level targeting see:

  Ben S. Bernanke (2017) Monetary Policy in a New Era, Brookings Institution, October 2, 2017.

Other Policies

Usually the weights on the different variables are found numerically:

- Solve for the law of motion that the central bank implements
- Calculate welfare for society
- Find the optimal λ_{CB} with a numerical solver or solve it in a grid.

Welfare Clarifications:

- Welfare under commitment is $< >$ than under discretion?
- Why?
- So what is the temptation to renege?

Notation:

- The path chosen at date T with commitment $\{y_t^{C_T}, \pi_t^{C_T}\}_{t=T}^{\infty}$
  - Important: for commitment we are distinguishing between the period of the allocation (t) and the period in which the path was designed (T).
  - At time $T = 0$ planner designs the plan. Inflation at period $t = 1$ in this path is given by $\pi_1^{C_0}$
  - At time $T = 1$ planner decides the plan. Inflation at period $t = 1$ from this path is given by $\pi_1^{C_1}$
- The path chosen with discretion $\{y_t^D, \pi_t^D\}_{t=0}^{\infty}$

Welfare Clarifications:

Problem from period $t = 0$:

Welfare of commitment policy is higher than that of the discretion policy.

$$-E_0 \sum_{t=0}^{\infty} \beta^t ((\pi_t^{C_0})^2 + \lambda (y_t^{C_0})^2) > -E_0 \sum_{t=0}^{\infty} \beta^t ((\pi_t^{D})^2 + \lambda (y_t^{D})^2)$$

Then at period $t = 1$:

But later on, promises start to be binding and the central bank would like to reoptimize:

$$-E_1 \sum_{t=1}^{\infty} \beta^{t-1} ((\pi_t^{C_0})^2 + \lambda (y_t^{C_0})^2) < -E_1 \sum_{t=1}^{\infty} \beta^{t-1} (\pi_t^{C_1})^2 + \lambda (y_t^{C_1})^2)$$

At a later date, commitment may be worse than discretion:

$$-E_1 \sum_{t=1}^{\infty} \beta^{t-1} ((\pi_t^{C_0})^2 + \lambda (y_t^{C_0})^2) \;???\; -E_1 \sum_{t=1}^{\infty} \beta^{t-1} ((\pi_t^{D})^2 + \lambda (y_t^{D})^2)$$

Imperfect Credibility

Two common ways to address optimal policy:

- Commitment (Time-inconsistent)
- Discretion (Time-consistent)

	Commitment	Discretion
Policy plan covers	Entire future	Nothing beyond current period
Reoptimizations	Never	Always
Ability to make promises	Perfect	Inexistent

Some reasons for imperfect credibility:

1. time-varying composition of monetary policy committees and central bank staff,
2. outside pressures of varying intensity,
3. economic research,
4. unforeseen events.

Model: imperfect credibility

Planner makes state-contingent promises regarding the entire future.

At the beginning of each time period, the occurrence of a re-optimization is driven by a two-state Markov stochastic process

$$x_t = \begin{cases} 1 & \text{with Prob. } \eta \\ 0 & \text{with Prob. } 1 - \eta \end{cases}$$

with $0 \leq \eta \leq 1$

- $x_t = 1$ previous promises are honored and the previous plan is continued.
- $x_t = 0$ previous promises are not honored and a new plan regarding the future is made.

Model: imperfect credibility

Commitment:

$$V(u_t, g_t) = \max_{\{y_t,\pi_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \{-(\pi_t^2 + \lambda y_t{}^2)\}$$

$$\text{s.t. } \pi_t = \kappa y_t + \beta E_t \pi_{t+1} + u_t$$

Discretion:

$$V^D(u_t, g_t) = \max_{\{y_t,\pi_t\}} E_0 \left\{-(\pi_t^2 + \lambda y_t{}^2) + \beta E_t V^D(u_{t+1}, g_{t+1})\right\}$$

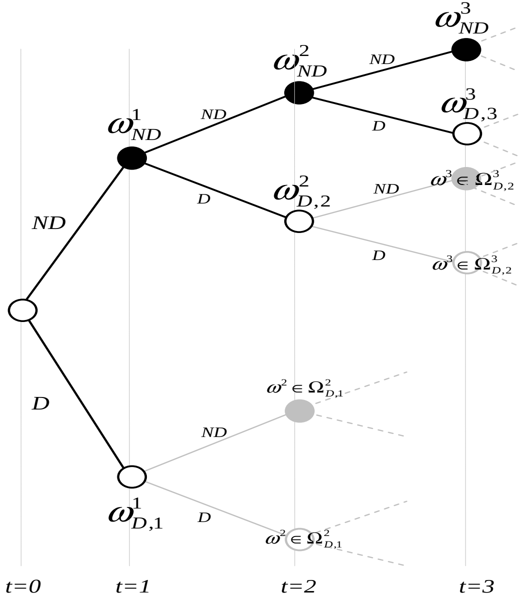
$$\text{s.t. } \pi_t = \kappa y_t + \beta E_t \Psi\left(u_{t+1}, g_{t+1}\right) + u_t$$

Imperfect Credibility:

$$V^D(u_t, g_t) = \max_{\{y_t,\pi_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} (\beta\eta)^t \{-(\pi_t^2 + \lambda y_t{}^2) + \beta\left(1-\eta\right) V^D(u_{t+1}, g_{t+1})\}$$

$$\text{s.t. } \pi_t = \kappa y_t + \beta\eta E_t \pi_{t+1} + \beta\left(1-\eta\right) E_t \Psi\left(u_{t+1}, g_{t+1}\right) + u_t$$

Model: imperfect credibility



Model: imperfect credibility

Denote $U_t = -(\pi_t^2 + \lambda y_t^2)$. Covering all the terms in the "tree":

$$V_t^D = U_t + \beta\eta U_{t+1} + \beta(1-\eta) V_{t+1}^D + \beta^2\eta^2 U_{t+2} + \beta^2\eta(1-\eta) V_{t+2}^D + \ldots \tag{3}$$

Rearranging:

$$V_t^D = U_t + \beta(1-\eta) V_{t+1}^D + \beta\eta\left\{U_{t+1} + \beta(1-\eta) V_{t+2}^D\right\} + + \beta^2\eta^2\left\{U_{t+2} + \beta(1-\eta) V_{t+3}^D\right\} + \ldots \tag{4}$$

Writing this as an infinite sum:

$$V_t^D = E_0 \sum_{t=0}^{\infty} (\beta\eta)^t \left\{U_t + \beta(1-\eta) V_{t+1}^D\right\} \tag{5}$$

Or alternatively (we will come back to this), consider the recursive representation and solve forward:

$$V_t^C(., \gamma_{t-1}) = U_t + \beta\eta V_t^C(., \gamma_t) + \beta(1-\eta) V_{t+1}^D \tag{6}$$

Model: imperfect credibility

- This approach cannot address the reasons of default, but can address the consequences.
- The central bank and private agents are aware and internalize reoptimizations.
- Analogous approach to Calvo-Yun pricing.
- Debortoli and Nunes (2010) show that results are similar when defaults are time-dependent.
- Simplicity is required for implementation in large scale models and this framework moves away from perfect commitment or no-commitment at all.

Some theory behind commitment solution

- In models without time-inconsistency, recursive programming techniques prove that a solution exists and is a function of certain variables (state-variables).

$$\max_{\{C_t, K_{t+1}, K_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, K_{t+1}, K_t, s_t)$$

$$s.t : K_{t+1} = g(C_t, K_t, s_t) \qquad t = 0, ...\infty$$

s_t exogenous and Markov

Some theory behind commitment solution

One can prove that a value function exists with certain properties:

$$V(K_0, s_0) = \max_{\{C_t, K_{t+1}, K_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t^*, K_{t+1}^*, K_t^*, s_t\right)$$

$$V(K_t, s_t) = U(C_t, K_{t+1}, K_t, s_t) + \beta E_t V(K_{t+1}, s_{t+1}).$$

One can prove that the optimal allocations C_t, K_{t+1}, K_t can be written as a time-invariant function ϕ that depends on the state variables:

$$\{C_t, K_{t+1}\} = \phi(K_t, s_t).$$

Some theory behind commitment solution

- These results are often forgotten because we use them without almost even noticing. We use these results when:
  - solving a model in any toolkit (Uhlig, Sims, Dynare). These solutions imply a guess and verify where we start with a function of state-variables.
  - solving for nonlinear models where we assume a functional form for the solution.
  - In both cases, we assume a policy function and can only assess that it converged according to some criteria (and always up to machine precision).

Some theory behind commitment solution

Some papers have extended the recursive programming techniques to problems with time-inconsistency.

$$V(u_t, g_t) = \max_{\{y_t,\pi_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \{-(\pi_t^2 + \lambda {y_t}^2)\}$$

$$\text{s.t. } \pi_t = \kappa y_t + \beta E_t \pi_{t+1} + u_t$$

Taking first order conditions:

$$\begin{aligned}
\gamma_t &: \pi_t - \kappa y_t - \beta E_t \pi_{t+1} - u_t = 0 \\
\pi_t &: -2\pi_t + \gamma_t - \gamma_{t-1} = 0 \\
y_t &: -2\lambda y_t - \kappa \gamma_t = 0 \\
\gamma_{-1} &= 0
\end{aligned}$$

The solution can be characterized as a function of the usual state-variables (in this case u_t) and the previous period lagrange multipliers associated with forward-looking constraints (γ_{t-1}).

Writing the commitment problem recursively:

$$V(u_t, g_t, \gamma_{t-1}) = \min_{\{\gamma_t\}} \max_{\{y_t, \pi_t\}} E_t\{-(\pi_t^2 + \lambda {y_t}^2) + \gamma_t (\pi_t - \kappa y_t - u_t) - \gamma_{t-1}\pi_t\}$$
$$+ \beta V(u_{t+1}, g_{t+1}, \gamma_t)$$
$$\gamma_{-1} = 0$$

Min Max formulation is only useful to prove some theorems. The bottom line is:

- The lagged lagrange multipliers associated with forward looking constraints need to be included as state variables.
- There exists value function: $V(u_t, g_t, \gamma_{t-1})$
- There exists an optimal policy function: $\psi(u_t, g_t, \gamma_{t-1})$

Imperfect Credibility:

$$V^D(u_t, g_t) = \max_{\{y_t,\pi_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} (\beta\eta)^t \left\{-(\pi_t^2 + \lambda y_t^2) + \beta(1-\eta) V^D(u_{t+1}, g_{t+1})\right\}$$

$$\text{s.t. } \pi_t = \kappa y_t + \beta\eta E_t \pi_{t+1} + \beta(1-\eta) E_t \Psi(u_{t+1}, g_{t+1}) + u_t$$

Map this problem into a regular commitment problem

- For given V^D and Ψ, this formulation maps directly into usual commitment problems.
- Discount factor is $\beta\eta$
- In the objective function, we have an infinite sum covering all the commitment terms.
- We have forward looking terms in the constraints.

Imperfect Credibility:

Writing the imperfect credibility problem recursively:

$$
\begin{aligned}
V(u_t, g_t, \gamma_{t-1}) = \min_{\{\gamma_t\}} \max_{\{y_t, \pi_t\}} & E_0\{-(\pi_t^2 + \lambda {y_t}^2) \\
& + \beta\eta V(u_{t+1}, g_{t+1}, \gamma_t) + \beta(1-\eta) V^D(u_{t+1}, g_{t+1})\} \\
& + \gamma_t (\pi_t - \kappa y_t - \beta(1-\eta) E_t \Psi(u_{t+1}, g_{t+1}) - u_t) - \gamma_{t-1}\pi_t \\
& \gamma_{-1} = 0
\end{aligned}
$$

Then the regular proofs of commitment apply and we can establish the following:

1. The value function exists and is defined as $V(u_t, g_t, \gamma_{t-1})$ where the previous period lagrange multiplier is a state variable.
2. The optimal allocations solving this problem can be summarized by a time-invariant function: $(y_t, \pi_t) = \psi(u_t, g_t, \gamma_{t-1})$.

Imperfect Credibility:

Writing the imperfect credibility problem recursively:

$$
\begin{aligned}
V(u_t, g_t, \gamma_{t-1}) = \min_{\{\gamma_t\}} \max_{\{y_t, \pi_t\}} & E_0\{-(\pi_t^2 + \lambda y_t{}^2) \\
& + \beta\eta V(u_{t+1}, g_{t+1}, \gamma_t) + \beta(1-\eta) V^D(u_{t+1}, g_{t+1})\} \\
& + \gamma_t (\pi_t - \kappa y_t - \beta(1-\eta) E_t \Psi(u_{t+1}, g_{t+1}) - u_t) - \gamma_{t-1}\pi_t \\
& \gamma_{-1} = 0
\end{aligned}
$$

Then the regular proofs of commitment apply and we can establish the following:

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2. The optimal allocations solving this problem can be summarized by a time-invariant function: $(y_t, \pi_t) = \psi(u_t, g_t, \gamma_{t-1})$.

Note: we have still not defined the equilibrium. What is V^D and Ψ?

Solution – Equilibrium Definition

The equilibrium with imperfect commitment satisfies the following conditions:

1. Given $\{y_t^D, \pi_t^D\}_{t=0}^{\infty}$ and the value V^D, the path $\{y_t, \pi_t\}_{t=0}^{\infty}$ solves the problem of the central bank in sequence form.
2. The value function V^D is such that $V^D(u_t, g_t) = V(u_t, g_t, \gamma_{t-1} = 0)$ and V is defined by the recursive formulation of the central bank's problem.
3. Denote the optimal policy functions as $(y_t, \pi_t) = \psi(u_t, g_t, \gamma_{t-1})$. The pair $\left(y_t^D, \pi_t^D\right)$ satisfies the condition $\left(y_t^D, \pi_t^D\right) = \psi(u_t, g_t, 0)$.

Note: we abused notation earlier on and wrote $\pi_{t+1}^D = \Psi(u_{t+1}, g_{t+1})$.

Solution – Equilibrium Definition

- What this means is that when there is a default, the optimization problem re-starts again without binding promises.
- Hence the value function and the optimal functions are exactly the same, but taking into account that previous promises are reneged on.
- It's a problem that restarts itself but obviously unlike promises, the physical state variables remain...
- Note that we could have defined analogous but different problems:
  - We could assume different commitment conditions when a default occurs.
  - We could also incorporate political disagreement and different parties.

Literature

- Imperfect credibility setting: Roberds (1987), Schaumburgh and Tambalotti (2007), Debortoli and Nunes (2010).
- Full-commitment:
  - Optimal policy in linear quadratic: Currie and Levine (1993), Soderlind (1999).
  - Solution algorithms: Uhlig (1995), Klein (2000), Sims (2002).
- Discretion (Markov-perfect equilibria):
  - Optimal policy in linear quadratic: Backus and Driffill (1985), Soderlind (1999), Dennis (2007).
  - Solution algorithms: Krusell, Quadrini, and Rios-Rull (1997), Judd (2004), Klein, Krusell, and Rios-Rull (2008).

Road map

DONE:

- Time-inconsistent solution. ✓
- Time-consistent solution. ✓
- Central Bank Communication. ✓
- Designing monetary policy objectives. ✓
- Imperfect Credibility. ✓
- Some theory behind solutions. ✓

Road map

NEXT:

- General model.
- Toolkit.
- Application to the Smets and Wouters (2007) model:
  - What are the gains of achieving more credibility?
  - How does the possibility of future re-optimizations affect current outcomes and promises?
  - How does imperfect credibility affect the shock propagation, volatilities, and cross-correlations between relevant variables?
  - Does the policy response to some shocks require more commitment? At what stages?

Model: general form

Consider a general linear model

$$A_{-1}y_{t-1} + A_0 y_t + A_1 E_t y_{t+1} + Bv_t = 0, \qquad \forall t$$

where y_t is a vector of endogenous variables, v_t is exogenous with $Ev_t = Ev_t v_{t-j} = 0$, $Ev_t v_t' = \Sigma_v$.

Models with more lags and leads, lagged expectations, constants, and serially correlated shocks can be accommodated by expanding the y_t vector.

The policymaker is assumed to have a quadratic loss function

$$\sum_{t=0}^{\infty} \beta^t y_t' W y_t$$

Model: general form with imperfect credibility

The central bank's problem can be written as:

$$y'_{-1}Py_{-1} + d = \min_{\{y_t\}_{t=0}^{\infty}} E_{-1} \sum_{t=0}^{\infty} (\beta\eta)^t \left[y'_t W y_t + \beta(1-\eta)(y'_t P y_t + d) \right]$$

s.t.: $A_{-1}y_{t-1} + A_0 y_t + \eta A_1 E_t y_{t+1} + (1-\eta) A_1 E_t y^D_{t+1} + Bv_t = 0. \forall t \geq 0$

Analogy with earlier model:

- Value function when reoptimizing at time $t+1$: $(y'_t P y_t + d)$
- Using the Markov-perfect assumption:

$$E_t y^D_{t+1} = \tilde{H} y_t.$$

  - We considered $\pi^D_{t+1} = \Psi(u_{t+1}, g_{t+1})$ and $\frac{\partial \Psi(u_{t+1}, g_{t+1})}{\partial \pi_t} = 0$
  - But if $\pi^D_{t+1} = \Psi(u_{t+1}, g_{t+1}, \pi_t)$ then $\frac{\partial \Psi(u_{t+1}, g_{t+1}, \pi_t)}{\partial \pi_t} \neq 0$

Solving Imperfect Credibility

- The solution is a time-invariant policy function with the Lagrange multiplier vector γ_{t-1} as *co-states*.

$$\begin{bmatrix} y_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} H_{yy} & H_{y\gamma} \\ H_{\gamma y} & H_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \gamma_{t-1} \end{bmatrix} + \begin{bmatrix} G_y \\ G_\gamma \end{bmatrix} v_t,$$

where the matrices H and G depend on the unknown matrix $\tilde{H}$.

- When a re-optimization occurs in period t, $\gamma_{t-1} = 0$.
  - This feature is not imposed. The theory shows that the optimal policy functions have this characterization.

$$y_t^D = H_{yy} y_{t-1} + G_y v_t$$
$$E_t y_{t+1}^D = H_{yy} y_t$$

- Therefore:

$$H_{yy} = \tilde{H}.$$

Solving Imperfect Credibility – Lagrangean

Form the Lagrangean and take first order conditions

$$\mathcal{L} \equiv E_{-1} \sum_{t=0}^{\infty} (\beta\eta)^t \left\{ y_t' \left[W + (1-\eta)\beta P \right] y_t + \gamma_{t-1}' \beta^{-1} A_1 y_t \right.$$

$$\left. \gamma_t' \left[A_{-1} y_{t-1} + \left(A_0 + (1-\eta) A_1 \tilde{H} \right) y_t + B v_t \right] \right\}$$

$$\gamma_{-1} = 0$$

$\tilde{H}, y_{-1}$ given.

Recap on matrix differentiation

Consider the expression $(x'a)$, where (x) is a vector of variables and (a) is a vector of parameters of size $(n \times 1)$.

Example of $n = 2$:

$$x'a = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = x_1 a_1 + x_2 a_2$$

Note that $x'a$ is of dimension (1×1). This is the same as the Lagrangian, we are optimizing one objective function.

Matrix differentiation:

$$\frac{\partial}{\partial x}(x'a) = \frac{\partial}{\partial x}(x_1 a_1 + x_2 a_2) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

The "rule" is that when differentiating with respect to a vector the result is a vector. Hence, we have:

$$\frac{\partial}{\partial x}(x'a) = a$$

Recap on matrix differentiation (cont.)

Also note that since $x'a$ is of dimension (1×1) :

$$x'a = a'x$$

Hence we have:

$$\frac{\partial}{\partial x}\left(x'a\right) = a$$
$$\frac{\partial}{\partial x}\left(a'x\right) = a$$

Recap on matrix differentiation (cont.)

Finally:

$$\begin{aligned} x'Ax &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} = A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 A_{11} + x_2 A_{12} & x_1 A_{12} + x_2 A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= (x_1)^2 A_{11} + x_2 x_1 A_{12} + x_1 x_2 A_{12} + (x_2)^2 A_{22} \\ &= (x_1)^2 A_{11} + 2 x_2 x_1 A_{12} + (x_2)^2 A_{22} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x}\left((x_1)^2 A_{11} + 2x_2 x_1 A_{12} + (x_2)^2 A_{22}\right) &= \begin{bmatrix} 2x_1 A_{11} + 2x_2 A_{12} \\ 2x_1 A_{12} + 2x_2 A_{22} \end{bmatrix} \\ &= 2 \begin{bmatrix} A_{11} & A_{12} \\ A_{21} = A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Hence:

$$\frac{\partial}{\partial x}\left(x'Ax\right) = 2Ax$$

Solving Imperfect Credibility – FOCs

$$\mathcal{L} \equiv E_{-1} \sum_{t=0}^{\infty} (\beta\eta)^t \left\{ y_t' \left[W + (1-\eta)\beta P\right] y_t + \gamma_{t-1}' \beta^{-1} A_1 y_t \right.$$

$$\left. \gamma_t' \left[A_{-1} y_{t-1} + \left(A_0 + (1-\eta) A_1 \tilde{H} \right) y_t + B v_t \right] \right\}$$

$$\gamma_{-1} = 0$$

$\tilde{H}, y_{-1}$ given.

$$\frac{\partial \mathcal{L}}{\partial \gamma_t} = A_{-1} y_{t-1} + \left[A_0 + (1-\eta) A_1 \tilde{H} \right] y_t + \eta A_1 E_t y_{t+1} + B v_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial y_t} = 2 W y_t + 2\beta (1-\eta) P y_t + \left(A_0 + (1-\eta) A_1 \tilde{H} \right)' \gamma_t$$

$$+ \mathcal{I}_\eta \beta^{-1} A_1' \gamma_{t-1} + \beta\eta A_{-1}' E_t \gamma_{t+1} = 0.$$

Solving Imperfect Credibility – FOCs

$$\frac{\partial \mathcal{L}}{\partial \gamma_t} = A_{-1} y_{t-1} + [A_0 + (1-\eta) A_1 H_{yy}] y_t + \eta A_1 E_t y_{t+1} + B v_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial y_t} = 2 W y_t + \beta (1-\eta) A'_{-1} E_t \gamma^D_{t+1} + (A_0 + (1-\eta) A_1 H_{yy})' \gamma_t$$
$$+ \mathcal{I}_\eta \beta^{-1} A'_1 \gamma_{t-1} + \beta \eta A'_{-1} E_t \gamma_{t+1} = 0.$$

Three notes:

First, substitute $\tilde{H} = H_{yy}$.

Second, we have already substituted the envelope condition

$$\frac{\partial y'_{t-1} P y_{t-1}}{\partial y_{t-1}} = 2 P y_{t-1} = A'_{-1} \gamma^D_t \implies \frac{\partial y'_t P y_t}{\partial y_t} = 2 P y_t = A'_{-1} E_t \gamma^D_{t+1}.$$

Third, $\mathcal{I}_\eta = 0$ if $\eta = 0$ and $\mathcal{I}_\eta = 1$ if $\eta > 0$. Go

Solving Imperfect Credibility – rearranging FOCs

Use the law of motion:

$$\begin{bmatrix} y_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} H_{yy} & H_{y\gamma} \\ H_{\gamma y} & H_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \gamma_{t-1} \end{bmatrix} + \begin{bmatrix} G_y \\ G_\gamma \end{bmatrix} v_t$$

Substituting:

$$\begin{aligned} E_t y_{t+1} &= H_{yy} y_t + H_{y\gamma}\gamma_t \\ E_t \gamma_{t+1} &= H_{\gamma y} y_t + H_{\gamma\gamma}\gamma_t \\ E_t \gamma^D_{t+1} &= H_{\gamma y} y_t, \end{aligned}$$

One obtains:

$$\Gamma_0 \equiv \begin{bmatrix} A_0 + A_1 H_{yy} & \eta A_1 H_{y\gamma} \\ 2W + \beta A'_{-1} H_{\gamma y} & A'_0 + (1-\eta) H'_{yy} A'_1 + \beta\eta A'_{-1} H_{\gamma\gamma} \end{bmatrix}$$

$$\Gamma_1 \equiv \begin{bmatrix} A_{-1} & 0 \\ 0 & \beta^{-1}\mathcal{I}_\eta A'_1 \end{bmatrix}, \qquad \Gamma_v \equiv \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

Solving Imperfect Credibility – rearranging FOCs

Hence we arrive at:

$$\Gamma_0 \begin{bmatrix} y_t \\ \gamma_t \end{bmatrix} + \Gamma_1 \begin{bmatrix} y_{t-1} \\ \gamma_{t-1} \end{bmatrix} + \Gamma_v v_t = 0,$$

The resulting law of motion is

$$\begin{bmatrix} y_t \\ \gamma_t \end{bmatrix} = \underbrace{-\Gamma_0^{-1}\Gamma_1}_{H} \begin{bmatrix} y_{t-1} \\ \gamma_{t-1} \end{bmatrix} \underbrace{-\Gamma_0^{-1}\Gamma_v}_{G} v_t,$$

We started with a law of motion of this type. But the H and G that we started with are not the same as this one.

Solving Imperfect Credibility – Iterative Algorithm

In summary, the algorithm proceeds as follows:

1. Using a guess H_{guess}, form Γ_0 and Γ_1.
2. Compute $H = -\Gamma_0^{-1}\Gamma_1$.
3. Check if $||H - H_{guess}|| < \xi$, where $||.||$ is a distance measure and $\xi > 0$. If the guess and the solution have converged, proceed to step 4. Otherwise, update the guess as $H_{guess} = H$ and repeat steps 1-3 until convergence.
4. Finally, form Γ_v and compute $G = -\Gamma_0^{-1}\Gamma_v$.

Clearly, there are alternative algorithms to the one proposed. For a given H the system of equations could be solved: Go

- using a generalized Schur decomposition as in Blanchard and Kahn (1980),
- solving a quadratic matrix equation as in Uhlig (1995),
- using a Newton-type method to find H.

Imperfect Credibility – Welfare

One can also determine matrix formulas for conditional and unconditional welfare. Go

Application: Smets and Wouters (2007) model

- Nominal frictions – sticky price and wage settings allowing for backward inflation indexation.
- Real rigidities – habit formation in consumption, investment adjustment costs, variable capital utilization, and fixed costs in production.
- Six orthogonal shocks: total factor productivity, two shocks affecting the intertemporal margin (risk premium and investment-specific technology shocks), two shocks affecting the intratemporal margin (wage and price-markup shocks), and an exogenous government spending shock.

Detailed Model Description Calibration

Application: Smets and Wouters (2007) model

- All parameters are calibrated to the posterior mode as reported in Smets and Wouters (2007). Calibration
- No interest rate rule nor the associated monetary policy shock. Instead, the central bank solves an optimal policy problem.
- The benchmark formulation is given by

$$U_t^b = w_\pi \pi_t^2 + w_y y_t^2 + w_i^b (i_t - i_{t-1})^2,$$

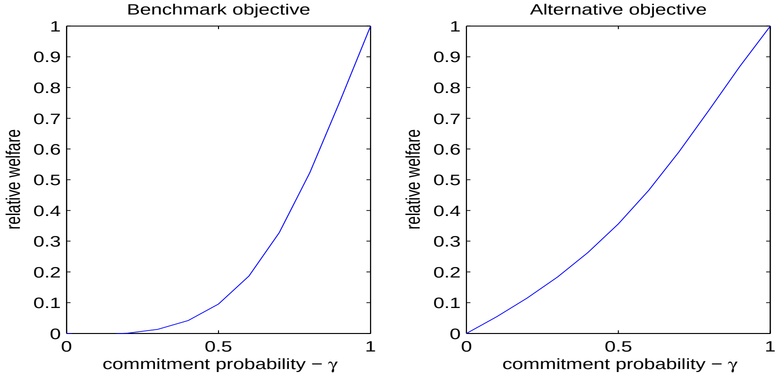
while the alternative specifications takes the form

$$U_t^a = w_\pi \pi_t^2 + w_y y_t^2 + w_i^a i_t^2.$$

Following Woodford (2003), $w_\pi = 1$, $w_y = 0.003$, $w_i^b = 0.0176$, and $w_i^a = 0.0048$.

Application: relative welfare

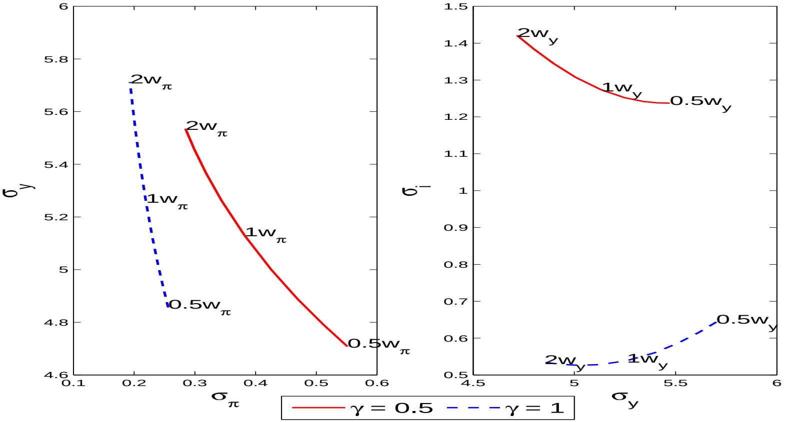
Relative welfare is defined as $(V_\eta - V_{\eta=0})/(V_{\eta=1} - V_{\eta=0})$



Application: Credibility and volatilities

Higher credibility $\rightarrow$ better management of the policy trade-offs because forward guidance is more effective as a policy tool.

Does higher credibility translate to lower volatilities of all welfare relevant variables?



Application: Credibility and simple interest rate rules

- The optimal policy can be implemented through targeting or interest rate rules.
- In DSGE models it is common to model the central bank's behavior through simple reduced-form interest rate rules. Clearly, such behavior is affected by the degree of commitment η.
- How are changes in η captured by the parameters of a simple rule?
- We perform a Monte-Carlo exercise taking our optimal policy model as the pseudo-true data generating process and estimate the rule

$$i_t = \phi_i i_{t-1} + \phi_\pi \pi_t + \phi_y y_t + \epsilon_t,$$

Application: Credibility and simple interest rate rules

	Benchmark Loss Function				Alternative Loss Function				U.S. Data
	1	0.9	0.5	0	1	0.9	0.5	0	(1970-2008)
ϕ_π	0.241 (0.047)	0.207 (0.103)	1.204 (0.141)	1.914 (0.048)	0.175 (0.043)	0.057 (0.138)	0.725 (0.312)	2.334 (0.072)	0.128 (0.039)
ϕ_y	0.002 (0.003)	-0.003 (0.007)	0.059 (0.014)	0.105 (0.005)	0.002 (0.002)	-0.010 (0.009)	-0.030 (0.033)	0.12 (0.008)	0.042 (0.009)
ϕ_i	0.971 (0.022)	0.926 (0.033)	0.875 (0.038)	0.75 (0.015)	0.972 (0.022)	0.843 (0.06)	0.503 (0.062)	0.159 (0.027)	0.926 (0.028)
R^2	0.923	0.865	0.843	0.977	0.921	0.759	0.416	0.930	0.947

- Some Features:
  - ϕ_i increases with η,
  - R^2 is higher for η close to 0 and 1.
- Plausibility of $\eta = 0.9$:
  - $\phi_y \approx 0$,
  - ϕ_i is high,
  - ϕ_π is plausible.
- Coefficients in simple interest rate rules may change even though preferences have not.

IRFs with imperfect commitment

For given initial conditions y_{-1}, γ_{-1}, and histories of the shocks $\{v_t, x_t\}_{t=0}^{T}$, the model simulation follows the formula

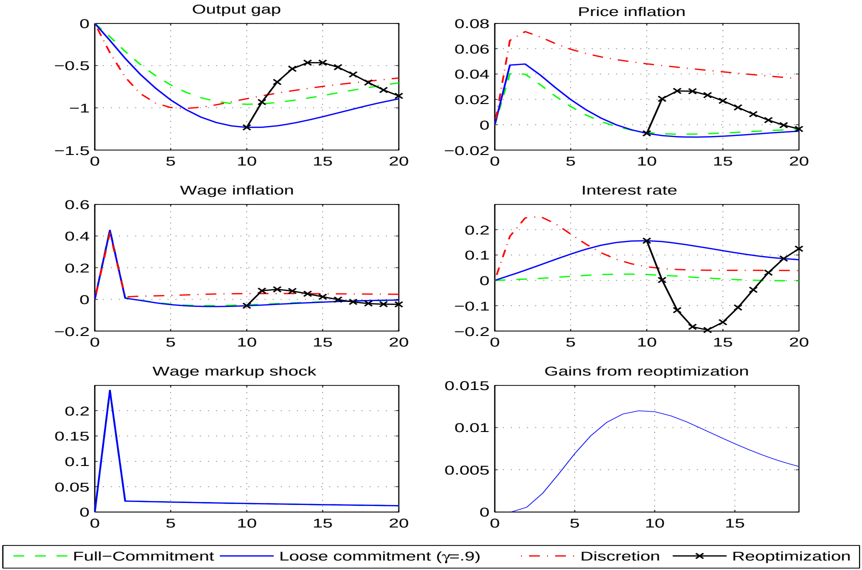
$$\begin{bmatrix} y_t \\ \gamma_t \end{bmatrix} = H \begin{bmatrix} y_{t-1} \\ x_t \gamma_{t-1} \end{bmatrix} + G v_t .$$

A history of the shock driving the re-optimizations (x_t) should also be specified.

Some interesting histories:

1. $x_t = 1 \forall t$ – an important history to influence expectations.
2. $x_t = 1 \forall t \neq m, \ \ x_m = 0$ – effects of reoptimizing in period m).
3. Averages across many histories of $\{x_t\}_{t=0}^{T}$ – average path.

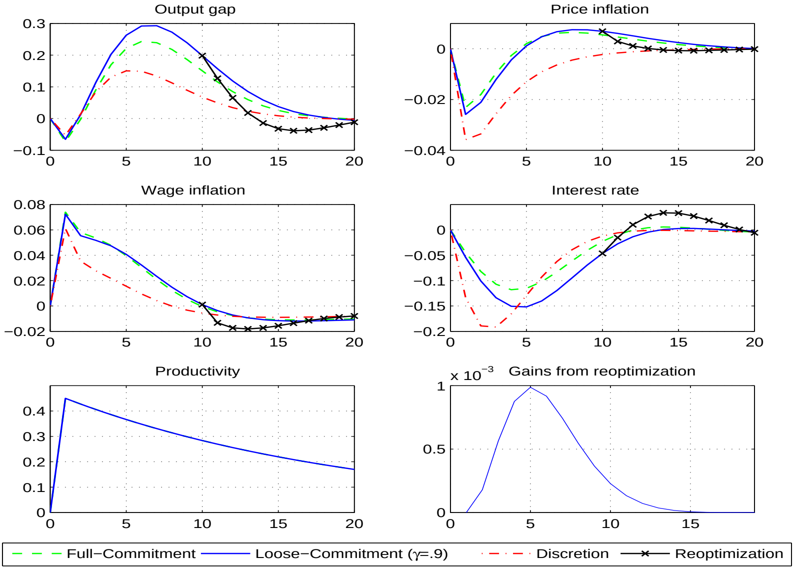
Application: IRFs, $\eta = .9$, wage markup shock



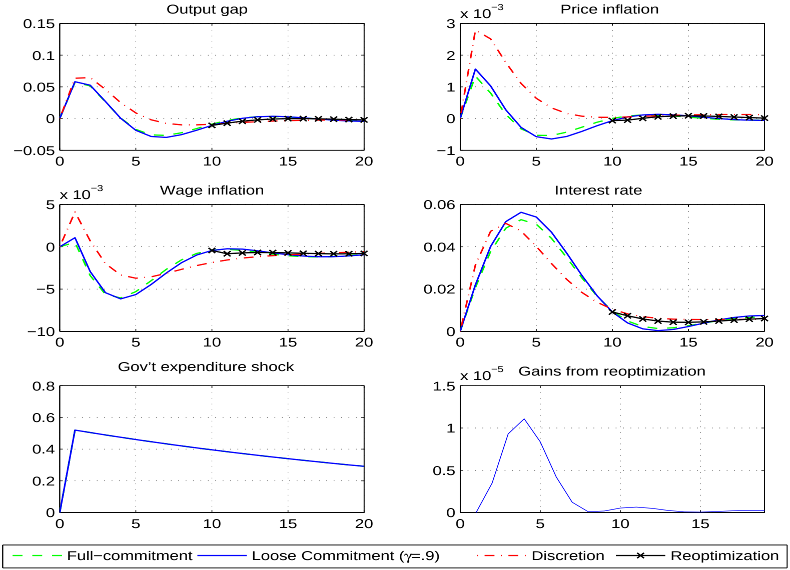
Application: IRFs, $\eta = .9$, wage markup shock

- Imperfect credibility is NOT the average of discretion and commitment. The response of the nominal interest rate does not lie between full-commitment and discretion.
  - The uncertainty about future reoptimizations affects the path where reoptimizations do not occur.
- Optimal response implies promising a deeper recession and low inflation.
  - With loose commitment, the interest rate needs to be higher because of expectations of future reoptimizations and associated reductions in interest rates.
- If a reoptimization occurs these promises are abandoned: inflation and output gap are increased through a reduction in the interest rate.
- Welfare gains of reoptimization are higher roughly after 9 quarters (both inflation and output gap are below target.)
- Commitment seems to be more important for markup shocks than productivity or demand shocks.

Application: IRFs, $\eta = .9$, productivity shock



Application: IRFs, $\eta = .9$, gov. spending shock



Application: Effects on second moments

	Model				U.S. Data
	Full-Com.	Loose Commitment		Discr.	(1970 - 2008)
		0.9	0.5		
Standard deviation (w.r.t. output)					
Output-gap	0.83	0.84	0.83	0.83	0.74
Price inflation	0.04	0.04	0.06	0.07	0.21
Wage inflation	0.08	0.08	0.08	0.09	0.26
Interest rate	0.09	0.15	0.21	0.18	0.29
Cross-correlations with output					
Output-gap	0.87	0.88	0.86	0.86	0.90
Price inflation	0.05	-0.17	-0.66	-0.70	-0.13
Wage inflation	0.21	0.13	-0.29	-0.38	0.05
Interest rate	-0.34	-0.49	-0.56	-0.56	-0.32

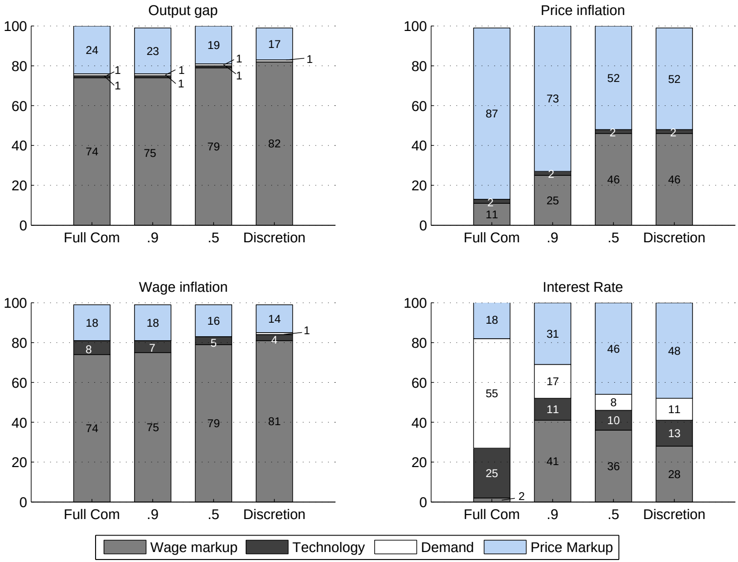
Some observations, when $\eta \downarrow$:

- 4th row: $\sigma(i) \uparrow$ (because less commitment means less persistence).
- 6th row: $\rho(out., \pi) \downarrow$ (because of markup shocks).

We are not estimating parameters but ... for $\eta = 0.9$:

- 6th row: $\rho(out., \pi) <\approx 0$
- 7th row: $\rho(out., \pi_w) >\approx 0$
- 4th row: relative $\sigma(i)$.

Application: variance decompositions



Application: variance decompositions

Patterns:

- Interest rate:
  - Demand shocks: $\eta\downarrow$ does not affect much the volatility of i_t. Go
  - Markup shocks: $\eta\downarrow$ increases the volatility of i_t. Go
- All variables:
  - $\eta\downarrow$ the importance of wage markup shocks increases.
  - This is the shock that requires relatively more commitment.

Closing remarks

- We hope you enjoyed this course and the CIMS Summer School.
- Hopefully these tools and concepts are of use to you in the future.
- Any feedback is much appreciated.
- We are more than happy to provide e-mail support on the course materials. Email us at: ricardo.nunes@surrey.ac.uk, d.park@surrey.ac.uk and l.rondina@sussex.ac.uk.

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Additional Material

Determining Unconditional Welfare:

We follow the steps in Marcet and Marimon (1998) to write the value function.

We add the constraints to the objective function, which does not alter its value, since in equilibrium the constraints equal zero.

The value of the objective function can thus be written as:

$$\sup_{\{\lambda_t\}_{t=0}^{\infty}} \inf_{\{y_t\}_{t=0}^{\infty}} E_{-1} \sum_{t=0}^{\infty} (\beta\gamma)^t \left[y_t' W y_t + \beta (1-\gamma) \left(y_t' P y_t + d \right) + \right. \tag{7}$$
$$\left. + \lambda_t' \left(\left(A_0 + (1-\gamma) A_1 H_{yy} \right) y_t + \gamma A_1 y_{t+1} + A_{-1} y_{t-1} + B v_t \right) \right]$$

Shifting the forward looking variables we have that:

$$\sup_{\{\lambda_t\}_{t=0}^{\infty}} \inf_{\{y_t\}_{t=0}^{\infty}} E_{-1} \sum_{t=0}^{\infty} (\beta\gamma)^t \left[y_t' W y_t + \beta (1-\gamma) \left(y_t' P y_t + d \right) + \right. \tag{8}$$

$$\left. + \lambda_t' \left(\left(A_0 + (1-\gamma) A_1 H_{yy} \right) y_t + A_{-1} y_{t-1} + B v_t \right) + \beta^{-1} \lambda_{t-1}' A_1 y_t \right]$$

$$\lambda_{-1}' = 0$$

For any initial condition $\left[\begin{array}{cc} y_{t-1}' & \lambda_{t-1}' \end{array} \right]$ the welfare measure, unconditional on the first realization of v_0, is given by

$$\left[\begin{array}{c} y_{t-1} \\ \lambda_{t-1} \end{array} \right]' \widehat{P} \left[\begin{array}{c} y_{t-1} \\ \lambda_{t-1} \end{array} \right] + d. \tag{9}$$

Unconditional welfare must take this form because:

1. y_{t-1} and λ_{t-1} are state variables
2. unconditional welfare cannot depend on the current shock
3. welfare is linear quadratic
4. A constant absorbs the effects of second moments.

Using the functional form for unconditional welfare, the previous problem can be written recursively as:

$$\begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \end{bmatrix}' \widehat{P} \begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \end{bmatrix} + d = \sup_{\lambda_t} \inf_{y_t} E_{t-1} \left[y_t' W y_t + \beta (1-\gamma) \left(\begin{bmatrix} y_t \\ 0 \end{bmatrix}' \widehat{P} \begin{bmatrix} y_t \\ 0 \end{bmatrix} + d \right) + \right. \tag{10}$$

$$+ \lambda_t' \left(\left(A_0 + (1-\gamma) A_1 H_{yy} \right) y_t + A_{-1} y_{t-1} + B v_t \right) + \beta^{-1} \lambda_{t-1}' A_1 y_t +$$

$$\left. + \beta\gamma \left(\begin{bmatrix} y_t \\ \lambda_t \end{bmatrix}' \widehat{P} \begin{bmatrix} y_t \\ \lambda_t \end{bmatrix} + d \right) \right]$$

In order to obtain the measure of unconditional welfare we need to determine the matrix $\widehat{P}$ and the scalar d.

The matrix $\widehat{P}$ can be obtained as a derivative of the value function with respect to $\left[\begin{array}{cc} y'_{t-1} & \lambda'_{t-1} \end{array}\right]$:

$$2\widehat{P}\left[\begin{array}{c} y_{t-1} \\ \lambda_{t-1} \end{array}\right] = E_{t-1}\left[\begin{array}{cc} 0 & A'_{-1} \\ \beta^{-1}A_1 & 0 \end{array}\right]\left[\begin{array}{c} y_t \\ \lambda_t \end{array}\right] = \tag{11}$$

$$= E_{t-1}\left[\begin{array}{cc} 0 & A'_{-1} \\ \beta^{-1}A_1 & 0 \end{array}\right]\left(H\left[\begin{array}{c} y_{t-1} \\ \lambda_{t-1} \end{array}\right] + Gv_t\right) = \tag{12}$$

$$= \left[\begin{array}{cc} 0 & A'_{-1} \\ \beta^{-1}A_1 & 0 \end{array}\right] H \left[\begin{array}{c} y_{t-1} \\ \lambda_{t-1} \end{array}\right],$$

This implies that:

$$\widehat{P} = \frac{1}{2}\left[\begin{array}{cc} 0 & A'_{-1} \\ \beta^{-1}A_1 & 0 \end{array}\right] H. \tag{13}$$

Notice that in the most pertinent case with initial conditions $\lambda_{t-1} = 0$ the only relevant term would be the upper left block of $\widehat{P}$, which equals $A'_{-1}H_{\lambda y}$.

The constant d can be obtained considering that the recursive problem can be written in matrix form as (terms in blue are from the constraints):

$$\begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \end{bmatrix}' \widehat{P} \begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \end{bmatrix} + d = \tag{14}$$

$$= E_{t-1} \left[\begin{array}{c} \begin{bmatrix} y_t \\ \lambda_t \end{bmatrix}' \widetilde{V} \begin{bmatrix} y_t \\ \lambda_t \end{bmatrix} \\ + \begin{bmatrix} y_t \\ \lambda_t \end{bmatrix}' \left(\begin{bmatrix} 0 & \beta^{-1} A_1' \\ A_{-1} & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} v_t \right) + \beta d \end{array} \right]$$

where $\widetilde{V} =$

$$\left(\begin{bmatrix} W & 0 \\ A_0 + (1-\gamma) A_1 H_{yy} & 0 \end{bmatrix} + \beta(1-\gamma) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}' \widehat{P} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \beta\gamma \widehat{P} \right)$$

Substituting the law of motion of $\left[\begin{array}{cc} y_t & \lambda_t \end{array} \right]$ we obtain:

$$\left[\begin{array}{c} y_{t-1} \\ \lambda_{t-1} \end{array} \right]' \widehat{P} \left[\begin{array}{c} y_{t-1} \\ \lambda_{t-1} \end{array} \right] + d = \tag{15}$$

$$= E_{t-1} \left[\begin{array}{c} \left(H \left[\begin{array}{c} y_{t-1} \\ \lambda_{t-1} \end{array} \right] + Gv_t \right)' \widetilde{V} \left(H \left[\begin{array}{c} y_{t-1} \\ \lambda_{t-1} \end{array} \right] + Gv_t \right) \\ + \left(H \left[\begin{array}{c} y_{t-1} \\ \lambda_{t-1} \end{array} \right] + Gv_t \right)' \left(\left[\begin{array}{cc} 0 & \beta^{-1} A_1' \\ A_{-1} & 0 \end{array} \right] \left[\begin{array}{c} y_{t-1} \\ \lambda_{t-1} \end{array} \right] + \left[\begin{array}{c} 0 \\ B \end{array} \right] v_t \right) + \beta d \end{array} \right]$$

Focusing only on the constant d and noting that $E_{t-1} v_t = 0$, we must have:

$$d = E_{t-1} v_t' G' \left(\widetilde{V} G + \left[\begin{array}{c} 0 \\ B \end{array} \right] \right) v_t + \beta d \tag{16}$$

Now apply the rule of the trace $tr(ABC) = tr(CAB) = tr(BCA)$ and rearrange:

$$d = \frac{1}{1-\beta} tr \left[\Sigma_v \left(G' \widetilde{V} G + G' \left[\begin{array}{c} 0 \\ B \end{array} \right] \right) \right]. \tag{17}$$

Determining Conditional Welfare

We can now compute the conditional welfare, which is defined as follows:

$$\begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \\ v_t \end{bmatrix}' \widetilde{P} \begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \\ v_t \end{bmatrix} + \widetilde{d} = y_t' W y_t + \beta (1-\gamma) E_t \left(\begin{bmatrix} y_t \\ 0 \\ v_{t+1} \end{bmatrix}' \widetilde{P} \begin{bmatrix} y_t \\ 0 \\ v_{t+1} \end{bmatrix} + \widetilde{d} \right) \quad (18)$$
$$+ \lambda_t' \left((A_0 + (1-\gamma) A_1 H_{yy}) y_t + \lambda_t' A_{-1} y_{t-1} + \lambda_t' B v_t \right) + \beta^{-1} \lambda_{t-1}' A_1 y_t$$
$$+ \beta\gamma E_t \left(\begin{bmatrix} y_t \\ \lambda_t \\ v_{t+1} \end{bmatrix}' \widetilde{P} \begin{bmatrix} y_t \\ \lambda_t \\ v_{t+1} \end{bmatrix} + \widetilde{d} \right)$$

The expectation of conditional welfare must be unconditional welfare:

$$E_t \left(\begin{bmatrix} y_t \\ \lambda_t \\ v_{t+1} \end{bmatrix}' \widetilde{P} \begin{bmatrix} y_t \\ \lambda_t \\ v_{t+1} \end{bmatrix} + \widetilde{d} \right) = \left(\begin{bmatrix} y_t \\ \lambda_t \end{bmatrix}' \widehat{P} \begin{bmatrix} y_t \\ \lambda_t \end{bmatrix} + d \right)$$

Determining Conditional Welfare

We can now compute the conditional welfare, which is defined as follows:

$$\begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \\ v_t \end{bmatrix}' \widetilde{P} \begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \\ v_t \end{bmatrix} + \widetilde{d} = y_t' W y_t + \beta (1-\gamma) E_t \left(\begin{bmatrix} y_t \\ 0 \\ v_{t+1} \end{bmatrix}' \widetilde{P} \begin{bmatrix} y_t \\ 0 \\ v_{t+1} \end{bmatrix} + \widetilde{d} \right) \tag{18}$$

$$+ \lambda_t' \left((A_0 + (1-\gamma) A_1 H_{yy}) y_t + \lambda_t' A_{-1} y_{t-1} + \lambda_t' B v_t \right) + \beta^{-1} \lambda_{t-1}' A_1 y_t$$

$$+ \beta\gamma E_t \left(\begin{bmatrix} y_t \\ \lambda_t \\ v_{t+1} \end{bmatrix}' \widetilde{P} \begin{bmatrix} y_t \\ \lambda_t \\ v_{t+1} \end{bmatrix} + \widetilde{d} \right)$$

The expectation of conditional welfare must be unconditional welfare:

$$E_t \left(\begin{bmatrix} y_t \\ \lambda_t \\ v_{t+1} \end{bmatrix}' \widetilde{P} \begin{bmatrix} y_t \\ \lambda_t \\ v_{t+1} \end{bmatrix} + \widetilde{d} \right) = \left(\begin{bmatrix} y_t \\ \lambda_t \end{bmatrix}' \widehat{P} \begin{bmatrix} y_t \\ \lambda_t \end{bmatrix} + d \right)$$

Substituting the terms in blue we obtain the next expression:

Using this relation to rearrange the expression for conditional welfare we obtain the expression:

$$\begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \\ v_t \end{bmatrix}' \widetilde{P} \begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \\ v_t \end{bmatrix} + \widetilde{d} = y_t' W y_t + \beta (1-\gamma) \left(\begin{bmatrix} y_t \\ 0 \end{bmatrix}' \widehat{P} \begin{bmatrix} y_t \\ 0 \end{bmatrix} + d \right) + \\ + \lambda_t' \left((A_0 + (1-\gamma) A_1 H_{yy}) y_t + \lambda_t' A_{-1} y_{t-1} + \lambda_t' B v_t \right) + \beta^{-1} \lambda_{t-1}' A_1 y_t + \\ + \beta\gamma \left(\begin{bmatrix} y_t \\ \lambda_t \end{bmatrix}' \widehat{P} \begin{bmatrix} y_t \\ \lambda_t \end{bmatrix} + d \right) \tag{19}$$

Writing the problem in matrix form, and substituting the law of motion for $\begin{bmatrix} y_t & \lambda_t \end{bmatrix}$ we obtain:

$$\begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \\ v_t \end{bmatrix}' \widetilde{P} \begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \\ v_t \end{bmatrix} + \widetilde{d} = \left(H \begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \end{bmatrix} + G v_t \right)' \widetilde{V} \left(H \begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \end{bmatrix} + G v_t \right) + \\ \left(H \begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \end{bmatrix} + G v_t \right)' \left(\begin{bmatrix} 0 & \beta^{-1} A_1' \\ A_{-1} & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \lambda_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} v_t \right) + \beta d \tag{20}$$

We can thus obtain conditional welfare, for any given initial condition, by just evaluating the right-hand side of this last expression.

- In these derivations we have computed welfare using the recursive formulation of the Lagrangean. As mentioned in class, that formulation is equivalent to the original problem only after imposing the initial condition $\lambda_{-1} = 0$.

- If one wants to evaluate the welfare according to the original formulation of equation $\sum_{t=0}^{\infty} \beta^t y_t' W y_t$, but for a different value of λ_{-1}, one needs to subtract $\lambda_{-1}\beta^{-1} A_1 E_{-1} y_0$ and $\lambda_{-1}\beta^{-1} A_1 y_0$ to equations (9) and (20), respectively.

SW model equations: part 1

— Aggregate resource constraint:

$$y_t = c_y c_t + i_y i_t + z_y z_t + \varepsilon_t^g \tag{21}$$

c_t consumption, i_t investment, z_t capital utilization rate, ε_t^g exogenous spending.

— The consumption Euler equation:

$$c_t = c_1 c_{t-1} + (1 - c_1) E_t c_{t+1} + c_2 (l_t - E_t l_{t+1}) - c_3 \left(r_t - E_t \pi_{t+1} + \varepsilon_t^b \right) \tag{22}$$

l_t hours worked, $(r_t - E_t \pi_{t+1})$ ex-ante real interest rate, ε_t^b finance shock.

— Investment equation:

$$i_t = i_1 i_{t-1} + (1 - i_1) E_t i_{t+1} + i_2 q_t + \varepsilon_t^i \tag{23}$$

q_t value of capital, ε_t^i investment specific shock.

SW model equations: part 2

— The corresponding arbitrage equation for the value of capital:

$$q_t = q_1 E_t q_{t+1} + (1 - q_1) E_t r^k_{t+1} - \left(r_t - E_t \pi_{t+1} + \varepsilon^b_t \right) \qquad (24)$$

r^k_{t+1} real rental rate on capital.

— Aggregate production function:

$$y_t = \phi_p \left(\alpha k^s_t + (1 - \alpha) l_t + \varepsilon^a_t \right) \qquad (25)$$

ϕ_p reflects the presence of fixed costs, k^s_t capital, ε^a_t productivity shock

— Installed capital and capital services:

$$k^s_t = k_{t-1} + z_t \qquad (26)$$

— Cost minimization by households links rental rate of capital and capital utilization

$$z_t = z_1 r^k_t \qquad (27)$$

SW model equations: part 3

— Accumulation of installed capital

$$k_t = k_1 k_{t-1} + (1 - k_1)\, i_t + k_2 \varepsilon_t^i \qquad (28)$$

— Cost minimization by firms defines the price mark-up (average price minus nominal marginal cost)

$$\mu_t^p = mpl_t - w_t = \alpha\,(k_t^s - l_t) + \varepsilon_t^a - w_t \qquad (29)$$

— New-Keynesian Phillips curve:

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \varepsilon_t^p \qquad (30)$$

— Rental rate of capital

$$r_t^k = -\,(k_t - l_t) + w_t \qquad (31)$$

SW model equations: part 4

— The wage mark-up

$$\mu_t^w = w_t - mrs_t = w_t - \left(\sigma_l l_t + \frac{1}{1-\lambda/\gamma}\left(c_t - \lambda/\gamma c_{t-1}\right)\right) \quad (32)$$

mrs_t marginal rate of substitution between working and consuming.

— Real wages adjust sluggishly

$$w_t = w_1 w_{t-1} + (1 - w_1)\left(E_t w_{t+1} + E_t \pi_{t+1}\right) - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu_t^w + \varepsilon_t^w \quad (33)$$

— Monetary policy reaction function:

$$r_t = \rho r_{t-1} + (1-\rho)\left[r_\pi \pi_t + r_y\left(y_t - y_t^p\right) + r_{\Delta y}\Delta\left(y_t - y_t^p\right)\right] + \varepsilon_t^r \quad (34)$$

Parametrization

Table A.1: Parameter Values in Smets and Wouters (2007).

Panel A: Calibrated

Parameter	Description	Value	Parameter	Description	Value
δ	Depreciation rate	0.025	ϵ_p	Kimball Elast. GM	10
ϕ_w	Gross wage markup	1.50	ϵ_w	Kimball Elast. LM	10
g_y	Gov't G/Y ss-ratio	0.18			

Panel B: Estimated

Parameter	Description	Value	Parameter	Description	Value
φ	Investment adj. cost	5.48	α	Capital production share	0.19
σ_c	Inv subs. elast. of cons.	1.39	ψ	Capital utilization cost	0.54
$\varkappa$	Degree of ext. habit	0.71	ϕ_p	Gross price markup	1.61
ξ_w	Calvo prob. wages	0.73	π	Steady state net infl. rate	0.0081
σ_l	Labor supply elas.	1.92	β	Discount factor	0.9984
ξ_p	Calvo prob. prices	0.65	$\bar{l}$	Steady state hours worked	0.25
ι_w	Ind. for non-opt. wages	0.59	γ	Steady state gross growth	1.0043
ι_p	Ind. for non-opt. prices	0.22			

Panel C: Shock Processes

Shock	Persistence		MA(1)		Std. of Innovation (%)	
Neutral Technology	ρ_a	0.95	-		σ_a	0.45
Risk premium	ρ_b	0.18	-		σ_b	0.24
Gov't spending	ρ_g	0.97	ρ_{ga}	0.52	σ_g	0.52
Inv. Specific Tech.	ρ_i	0.71			σ_i	0.45
Price markup	ρ_p	0.90	μ_p	0.74	σ_p	0.14
Wage markup	ρ_w	0.97	μ_w	0.88	σ_w	0.24
Monetary policy	ρ_r	-	-		σ_r	-

SW equations: additional details and description

The aggregate resource constraint:

$$y_t = c_y c_t + i_y i_t + z_y z_t + \varepsilon_t^g$$

Output (y_t) is absorbed by consumption (c_t), investment (i_t), capital-utilization costs that are a function of the capital utilization rate (z_t), and exogenous spending (ε_t^g);

Exogenous spending follows the process

$$\varepsilon_t^g = \rho_g \varepsilon_{t-1}^g + \eta_t^g + \rho_{ga} \varepsilon_t^a$$

The consumption Euler equation:

$$c_t = c_1 c_{t-1} + (1 - c_1) E_t c_{t+1} + c_2 (l_t - E_t l_{t+1}) - c_3 \left(r_t - E_t \pi_{t+1} + \varepsilon_t^b \right)$$

Current consumption (c_t) depends on a weighted average of past and expected future consumption, and on expected growth in hours worked ($l_t - E_t l_{t+1}$), the ex-ante real interest rate ($r_t - E_t \pi_{t+1}$), and a disturbance term ε_t^b.

The disturbance term ε_t^b represents a wedge between the interest rate controlled by the central bank and the return on assets held by the households. Under the assumption of no external habit formation and log utility in consumption, $c_1 = c_2 = 0$ and the traditional purely forward looking consumption equation is obtained.

Investment Euler equation:

$$i_t = i_1 i_{t-1} + (1 - i_1) E_t i_{t+1} + i_2 q_t + \varepsilon_t^i$$

where q_t is the real value of the existing capital stock, and ε_t^i represents a disturbance to the investment-specific technology process. The corresponding arbitrage equation for the value of capital is given by:

$$q_t = q_1 E_t q_{t+1} + (1 - q_1) E_t r_{t+1}^k - \left(r_t - E_t \pi_{t+1} + \varepsilon_t^b \right)$$

The current value of the capital stock (q_t) depends positively on its expected future value and the expected real rental rate on capital $\left(E_t r_{t+1}^k\right)$ and negatively on the ex-ante real interest rate and the risk premium disturbance.

The aggregate production function is given by

$$y_t = \phi_p \left(\alpha k_t^s + (1-\alpha)\, l_t + \varepsilon_t^a\right)$$

Output is produced using capital (k_t^s) and labor services (hours worked, l_t), ε_t^a denotes total factor productivity. The parameter α captures the share of capital in production, and the parameter ϕ_p is one plus the share of fixed costs in production.

$$k_t^s = k_{t-1} + z_t$$

As newly installed capital becomes effective only with a one-quarter lag, current capital services used in production (k_t^s) are a function of capital installed in the previous period (k_{t-1}) and the degree of capital utilization (z_t).

Cost minimization by the households that provide capital services implies that the degree of capital utilization is a positive function of the rental rate of capital,

$$z_t = z_1 r_t^k$$

The accumulation of installed capital (k_t) is a function not only of the flow of investment but also of the relative efficiency of these investment expenditures as captured by the investment specific technology disturbance

$$k_t = k_1 k_{t-1} + (1 - k_1)\, i_t + k_2 \varepsilon_t^i$$

Cost minimization by firms will also imply that the rental rate of capital is negatively related to the capital-labor ratio and positively to the real wage (both with unitary elasticity):

$$r_t^k = -(k_t^s - l_t) + w_t$$

Cost minimization by firms in monopolistic competitive goods market implies that the price mark-up (μ_t^p), defined as the difference between the average price and the nominal marginal cost or the negative of the real marginal cost, is equal to the difference between the marginal product of labor (mpl_t) and the real wage (w_t):

$$\mu_t^p = mpl_t - w_t = \alpha \left(k_t^s - l_t\right) + \varepsilon_t^a - w_t$$

The New Keynesian Phillips curve:

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \varepsilon_t^p$$

(π_t) depends positively on past and expected future inflation, negatively on the current price mark-up, and positively on a price mark-up disturbance (ε_t^p). The price mark-up disturbance is assumed to follow an ARMA(1, 1) process:

$$\varepsilon_t^p = \rho_p \varepsilon_{t-1}^p + \eta_t^p - \mu_p \eta_{t-1}^p.$$

In analogy with the goods market, in the monopolistically competitive labor market, the wage mark-up will be equal to the difference between the real wage and the marginal rate of substitution between working and consuming (mrs_t),

$$\mu_t^w = w_t - mrs_t = w_t - \left(\sigma_l l_t + \frac{1}{1 - \lambda/\gamma} \left(c_t - \lambda/\gamma c_{t-1} \right) \right)$$

where σ_l is the elasticity of labor supply with respect to the real wage, γ is the steady-state growth rate, and λ is the habit parameter in consumption.

Similarly, due to nominal wage stickiness and partial indexation of wages to inflation, real wages adjust only gradually to the desired wage mark-up:

$$w_t = w_1 w_{t-1} + (1 - w_1)(E_t w_{t+1} + E_t \pi_{t+1}) - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu_t^w + \varepsilon_t^w$$

The real wage w_t is a function of expected and past real wages, expected, current, and past inflation, the wage mark-up, and a wage mark-up disturbance (ε_t^w) assumed to follow an ARMA(1, 1) $\varepsilon_t^w = \rho_w \varepsilon_{t-1}^w + \eta_t^w - \mu_w \eta_{t-1}^w$.

The monetary policy rule (not used in the optimal policy)

$$\begin{aligned} r_t =& \rho r_{t-1} + (1 - \rho)\left[r_\pi \pi_t + r_y \left(y_t - y_t^p\right)\right] \\ & + r_{\Delta y}\left[\left(y_t - y_t^p\right) - \left(y_{t-1} - y_{t-1}^p\right)\right] + \varepsilon_t^D, \end{aligned}$$

where $\left(y_t - y_t^p\right)$ is the output-gap.

”They sang these words most musically, and as I longed to hear them further I made by frowning to my men that they should set me free; but they quickened their stroke, and Eurylochus and Perimedes bound me with still stronger bonds till we had got out of hearing of the Sirens’ voices. Then my men took the wax from their ears and unbound me.”

Book XII, Homer, The Odyssey, 8th century BC

Ulysses and the Sirens, J. W. Waterhouse (1849-1917), 1891