Introduction to Heterogeneous Agents and Risk Sharing

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Lecture 1

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Setup

- Lucas tree economy with stochastic labor income
- Preferences

$$\max E_0 \sum_{t=1}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \tag{1}$$

- Endowments: labor income

$$\begin{aligned} y_{it} &= \exp\left(z_{it}\right) \\ z_{it} &= z_{i,t-1} + \eta_{i,t} \\ \eta_{i,t} &\sim N\left(-\frac{\sigma^2}{2}, \sigma^2\right) \end{aligned}$$

Market structure and budget constraints

- Aggregate “production” is

$$Y_t = \sum_i y_{i,t}$$

- Recall that If $x \sim N\left(\mu, \sigma^2\right)$, then

$$E \exp\left(x\right) = \exp\left(\mu + \frac{\sigma^2}{2}\right).$$

Hence, as the population becomes large, $E \exp\left(\eta_{i,t}\right) = 1$, so that $Y_t = 1$ for all t.

Market structure and budget constraints

- Markets: there exists a Lucas tree in zero net supply paying interest rate r_t. Note: no insurance against endowment shock.
- Budget constraint

$$c_t + a_{t+1} = (1 + r_t)\, a_t + y_t \tag{2}$$

Equilibrium conditions

Definition

A sequential equilibrium is defined as an allocation $\{c_{it}, a_{it}\}_{i,t}$ and a set of prices $\{r_t\}$ such that

1. The allocation $\{c_{it}, a_{it}\}_{i,t}$ solves (1) subject to (2)
2. Market clearing:

$$
\begin{aligned}
C_t - Y_t &= \sum_i (c_{it} - y_{it}) = 0 \\
A_t &= \sum_i a_{it} = 0
\end{aligned}
$$

Guess and verify solution

- Individual optimization: Difficult to make progress beyond stating the FOC;

$$1 = \beta (1 + r) E_t \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \quad (3)$$

- Solution: guess autarky;

$$\begin{aligned} c_{it} &= y_{it} \\ a_{it} &= 0 \end{aligned}$$

Verify equilibrium conditions

- Market clearing is trivial
- Necessary to verify individual optimization: substitute the guess $c_{it} = y_{it}$ into the FOC (3)

$$
\begin{aligned}
1 &= \beta(1+r) E_t \left(\frac{y_{t+1}}{y_t}\right)^{-\gamma} \\
&= \beta(1+r) E_t \exp\left(-\gamma\left(z_{t+1} - z_t\right)\right) \\
&= \beta(1+r) E_t \exp\left(-\gamma\eta_t\right) \\
&= \beta(1+r) \cdot \exp\left(\frac{\gamma(1+\gamma)}{2}\sigma^2\right),
\end{aligned}
$$

where the last step follows from

$$
\begin{aligned}
E\left(-\gamma\eta_t\right) &= \gamma\frac{\sigma^2}{2} \\
var\left(-\gamma\eta_t\right) &= \gamma^2\sigma^2 \\
E_t \exp\left(-\gamma\eta_t\right) &= \exp\left(\gamma\frac{\sigma^2}{2} + \frac{\gamma^2\sigma^2}{2}\right) = \exp\left(\frac{\gamma(1+\gamma)}{2}\sigma^2\right).
\end{aligned}
$$

Conclusion

- autarky is an equilibrium if

$$1 + r_t = 1 + r = \frac{1}{\beta} \exp\left(-\frac{\gamma(1+\gamma)}{2}\sigma^2\right)$$

- If $\sigma > 0$ then $1 + r < 1/\beta$
- Lower r the larger is γ and the larger is σ (interest rate is lowered so that precautionary motive to save = intertemporal motive to dissave)

Demographics and preferences

- Preferences over sequences of consumption and hours worked:

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^{t}u\left(c_{t},h_{t};\varphi\right)$$

$$u\left(c_{t},h_{t};\varphi\right)=\frac{c_{t}^{1-\gamma}-1}{1-\gamma}-\exp\left(\varphi\right)\frac{h_{t}^{1+\sigma}}{1+\sigma}.$$

where φ is idiosyncratic disutility drawn from F_{φ}

Individual wages

Log individual wage is the sum of two orthogonal components:

$$\log w_t = \alpha_t + \varepsilon_t$$

- α follows a unit root process

$$\alpha_t = \alpha_{t-1} + \omega_t$$

  - where ω_t is drawn from $F_{\omega t}$
- ε i.i.d. drawn from F_ε, uncorrelated with α

Production and Government

- Aggregate production linear in aggregate effective labor
- Competitive markets: wages are individual productivities
- Government runs a progressive tax/transfer scheme to redistribute and to finance expenditure G_t
- Two parameter function (Feldstein 1969) maps pre-tax earnings ($y = wh$) to after-tax earnings ($\tilde{y}$):

$$\tilde{y} = \lambda y^{1-\tau}$$

- τ is the progressivity parameter

Private risk-sharing

1. Agents can save and borrow a risk-free bond b
   - Bonds in zero net supply
   - Agents enter with zero bonds
2. No explicit insurance against shocks to α
3. Full insurance against shocks to ε
   - Captures other insurance arrangements: financial markets, family, etc. plus pre-knowledge of future wage changes

Decentralizing Insurance

- Recall that α and ε are multiplicative in levels
  $\Rightarrow$ Want to scale insurance against shocks to ε to

  realization of shock to α

- Solution (possible since ε is serially uncorrelated)
  - first observe innovation to α
  - then buy insurance against ε

- Digression: if ε were persistence, the solution requires richer "island environment" where all inhabitants on an island have the same α and different permanent shocks to ε: $\Rightarrow$ ε can be insured within the island

Budget constraint (only transitory shocks to ε)

1. Beginning of period: innovation ω_t to α_t is realized
2. Middle of period: buy insurance against ε_t:

$$b_t = \int Q_t(\varepsilon) B_t(\varepsilon) d\varepsilon,$$

   where $Q_t(.)$ is price of insurance and $B_t(.)$ is quantity
3. End of period: ε_t is realized, consumption and labor supply chosen:

$$c_t + q_t b_{t+1} = \lambda (w_t h_t)^{1-\tau} + B_t(\varepsilon_t)$$

Equilibrium

- There is no bond trade in equilibrium

  $\Rightarrow$ Some shocks uninsured privately (α_t, φ), others perfectly insured (ε_t)

- Can solve for quantities and prices in closed-form

Connection to Constantinides and Duffie (1996)

- CRRA prefs, unit root shocks to log disposable income, zero initial wealth $\Rightarrow$ existence of a no trade equilibrium
- Our environment micro-founds unit root disposable income:

1. Start from richer process for individual wages
2. Labor supply: exogenous wages $\rightarrow$ endogenous earnings
3. Non-linear taxation: pre-tax earnings $\rightarrow$ after-tax earnings
4. Private risk sharing: earnings $\rightarrow$ gross income
5. No bond trade: disposable income = consumption

Hours worked

$$\log h_t^a(\varphi, \alpha, \varepsilon) = -\hat{\varphi} + \left(\frac{1-\gamma}{\widehat{\sigma}+\gamma}\right)\alpha + \frac{1}{\widehat{\sigma}}\varepsilon + \mathcal{H}_t^a$$

where $\widehat{\sigma} \equiv \frac{\sigma+\tau}{1-\tau}$ and $\hat{\varphi} \equiv \frac{\varphi}{\widehat{\sigma}+\gamma}$

- Response to ε given by tax-modified Frisch elasticity
- Response to α depends on value for γ which controls wealth effect

Consumption

$$\log c_t^a(\varphi, \alpha, \varepsilon) = -(1-\tau) \cdot \hat{\varphi} + (1-\tau) \cdot \left(\frac{1+\widehat{\sigma}}{\widehat{\sigma}+\gamma} \right) \alpha + \mathcal{C}_t^a$$

- Response to $(\alpha, \hat{\varphi})$ mediated by progressivity
- Invariant to insurable shock ε
- Note: Consumption follows a random walk, displays excess smoothness relative to PIH

Verify equilibrium

- Equilibrium conditions: market clearing and individual optimization
- Optimality conditions: first-order conditions:
  1. Euler equation for bond holdings

$$q = \mathbb{E}\left\{\beta \frac{u_c(t+1)}{u_c(t)}\right\}$$

  2. Intratemporal FOC for labor supply

$$-u_h(t) = \lambda \left[w \cdot u_c(t)\right]^{1-\tau}$$

Verify equilibrium

- Equilibrium conditions: market clearing and individual optimization
- Optimality conditions: first-order conditions:
  1. Euler equation for bond holdings

$$q = \mathbb{E}\left\{\beta \frac{\exp\left(-\gamma\left[-(1-\tau)\cdot\hat{\varphi} + (1-\tau)\cdot\left(\frac{1+\widehat{\sigma}}{\widehat{\sigma}+\gamma}\right)\ (\alpha_t + \omega_{t+1}) + \right.\right.}{\exp\left(-\gamma\left[-(1-\tau)\cdot\hat{\varphi} + (1-\tau)\cdot\left(\frac{1+\widehat{\sigma}}{\widehat{\sigma}+\gamma}\right)\ \alpha_t + \mathcal{C}^a\right]\right.}\right.$$

$$= \mathbb{E}\left\{\beta \exp\left(-\gamma(1-\tau)\cdot\left(\frac{1+\widehat{\sigma}}{\widehat{\sigma}+\gamma}\right)\ \omega_{t+1}\right)\right\}$$

  2. Intratemporal FOC for labor supply: verify using equlibrium allocations