Stability, Multiplicity, and Sunspots (deriving solutions to linearized system & Blanchard-Kahn conditions)

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Content

1. A lot on sunspots
2. A simple way to get policy rules in a linearized framework
    - and an even simpler way based on time iteration (an idea of Pontus Rendahl)

Introduction

- What do we mean with non-unique solutions?
  - multiple solution versus multiple steady states
- What are sunspots?
- Are models with sunspots scientific?

Terminology

- Definitions are very clear
  - (use in practice can be sloppy)

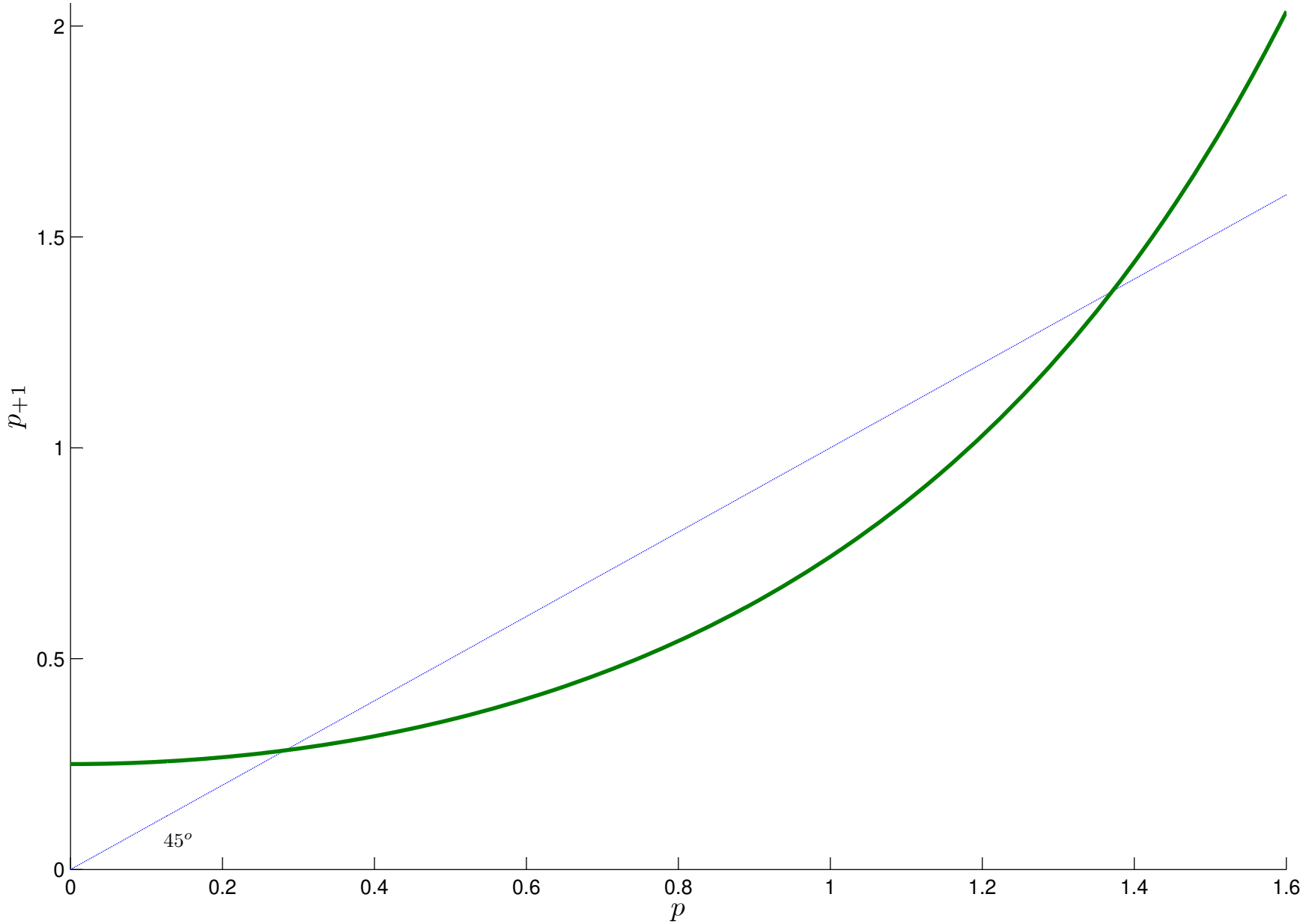
Model:

$$H(p_{+1}, p) = 0$$

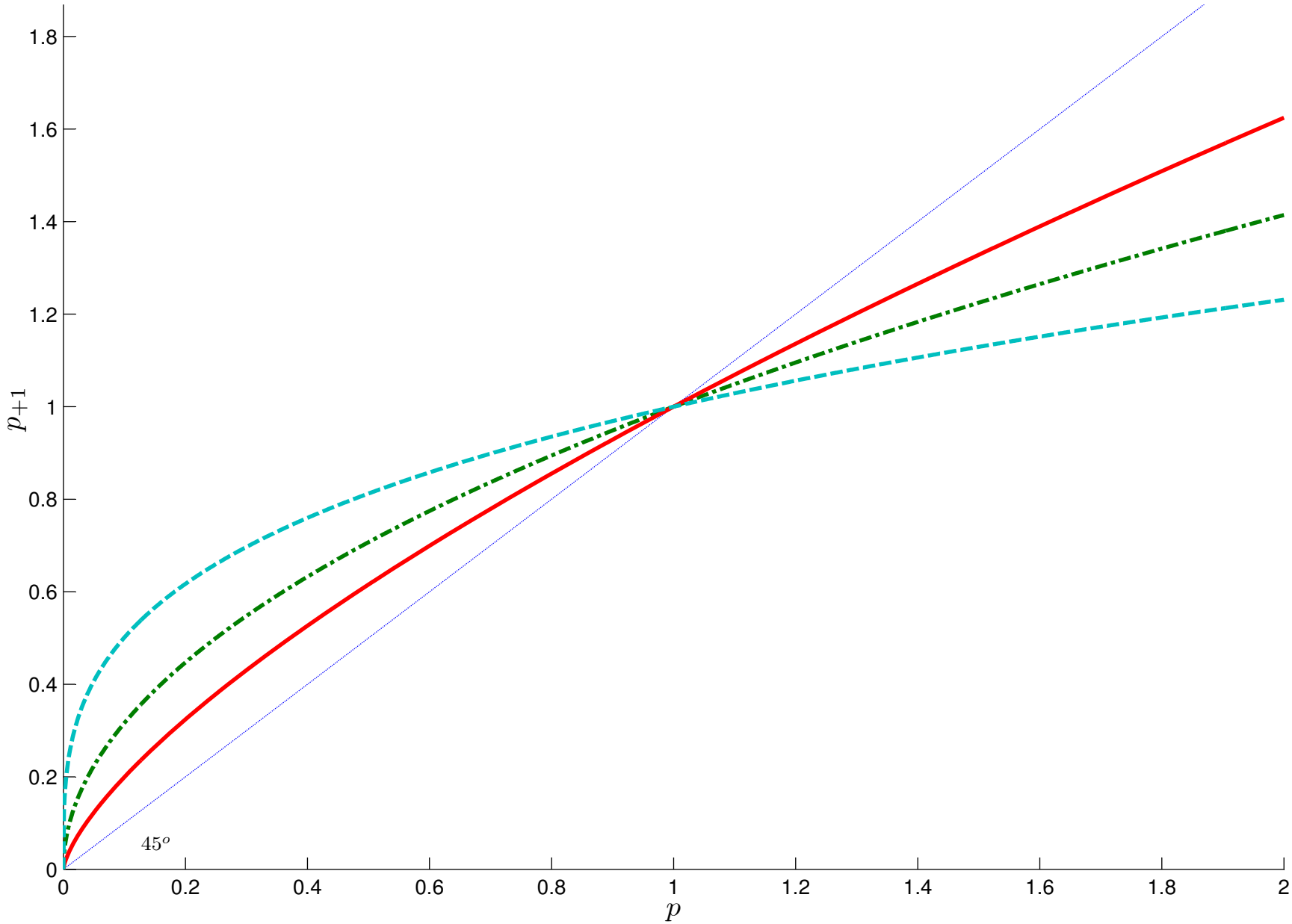
Solution:

$$p_{+1} = f(p)$$

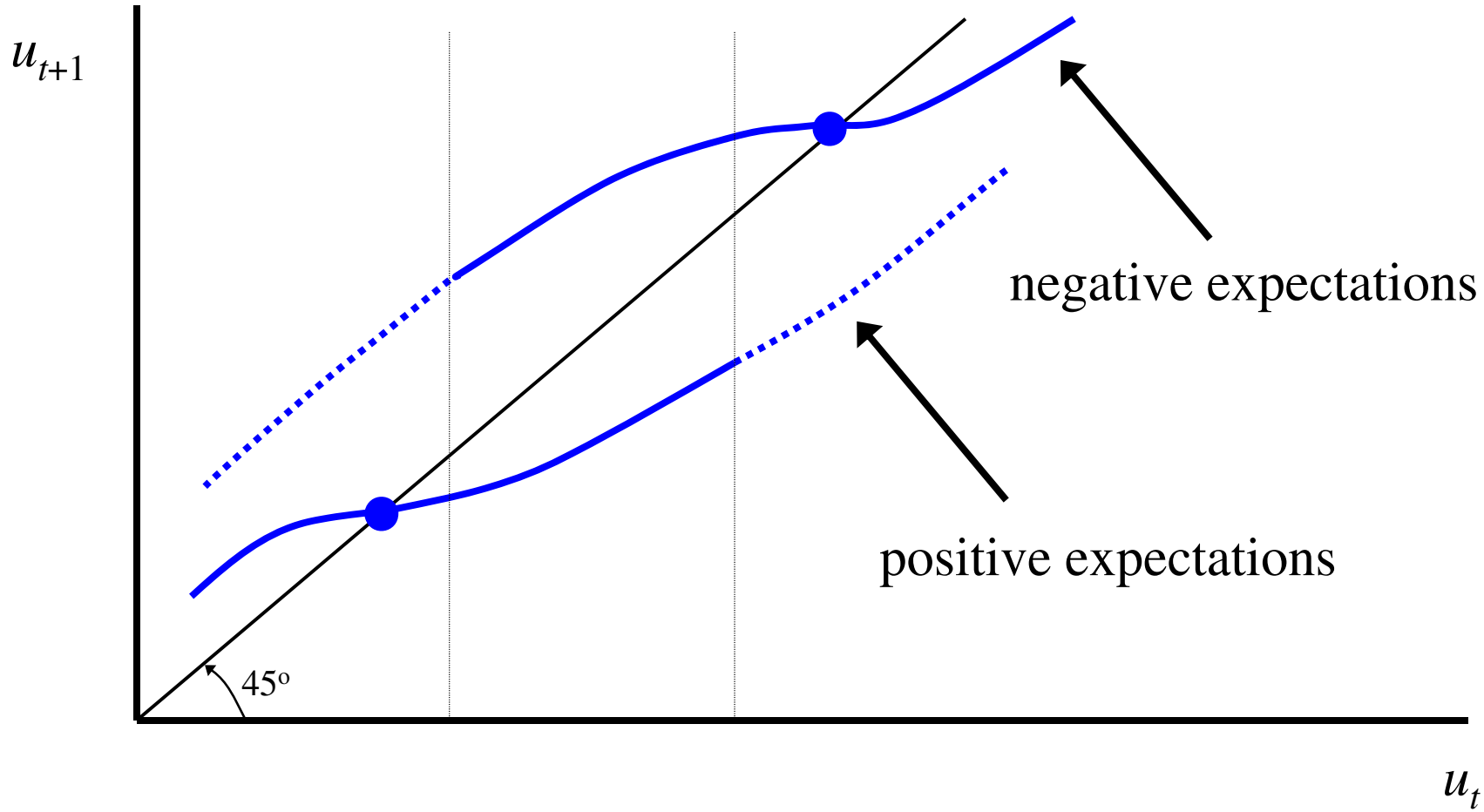
Unique solution & multiple steady states



Multiple solutions & unique (non-zero) steady state

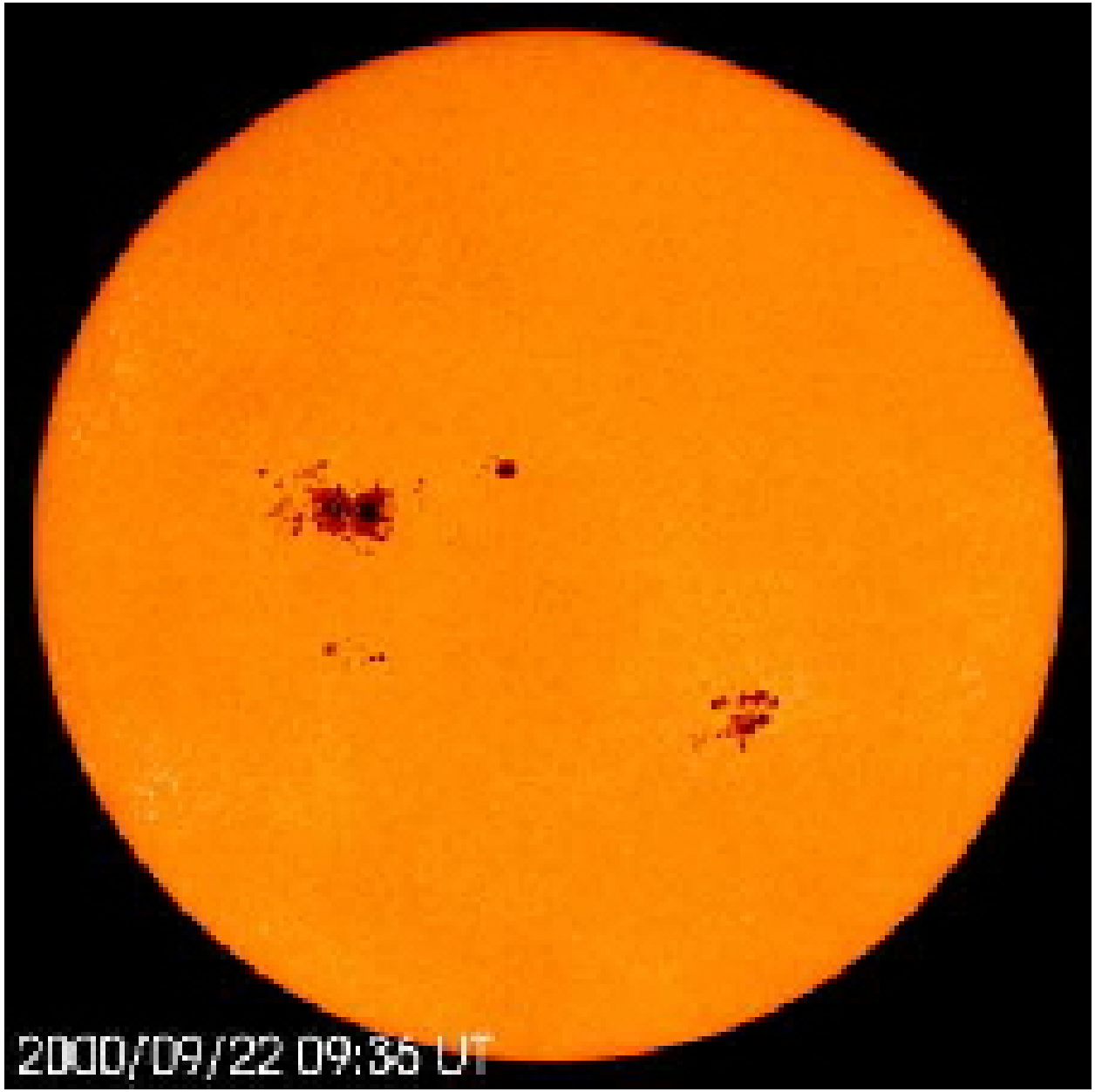


Multiple steady states & sometimes multiple solutions

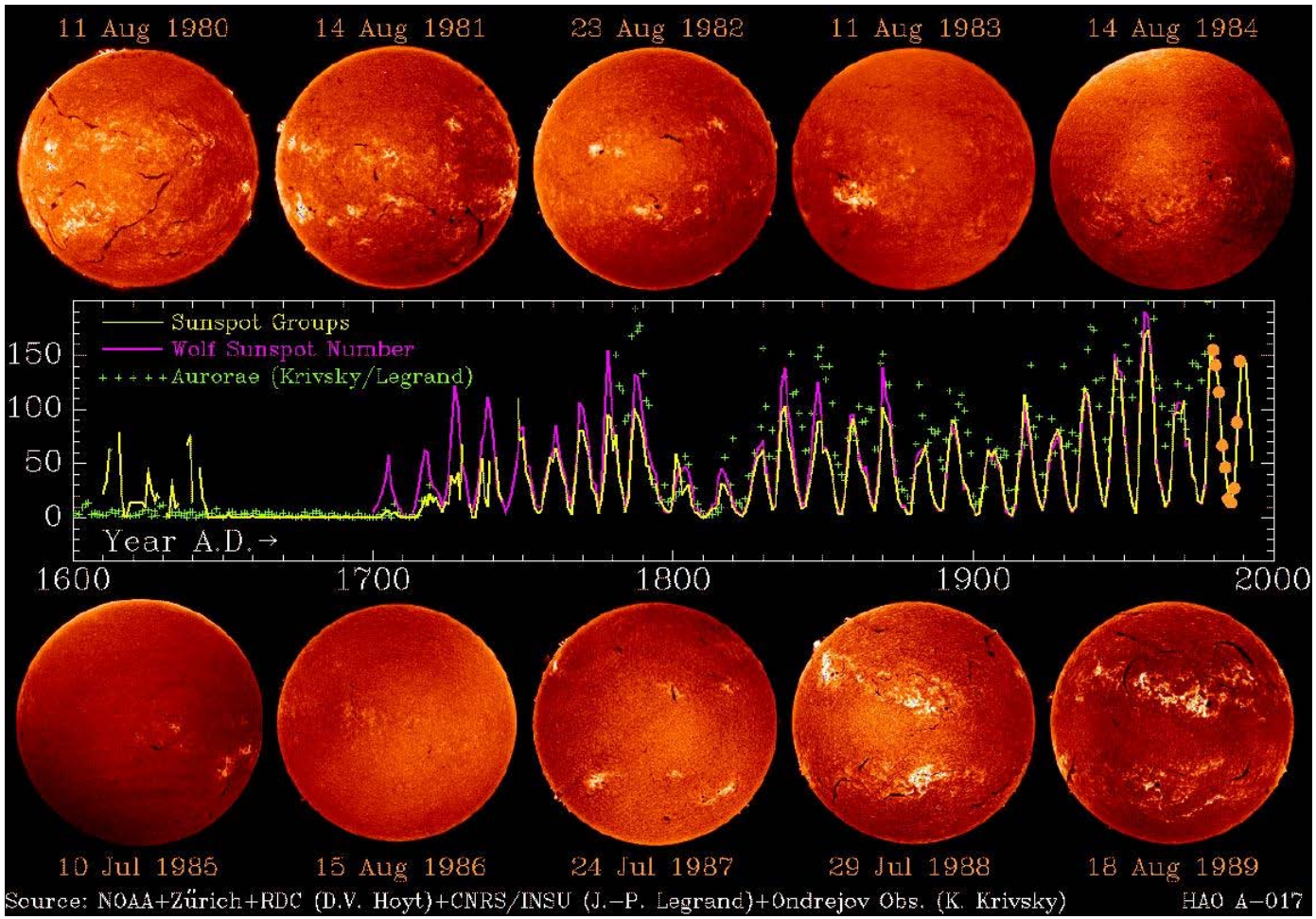


From Den Haan (2007)

Large sunspots (around 2000 at the peak)

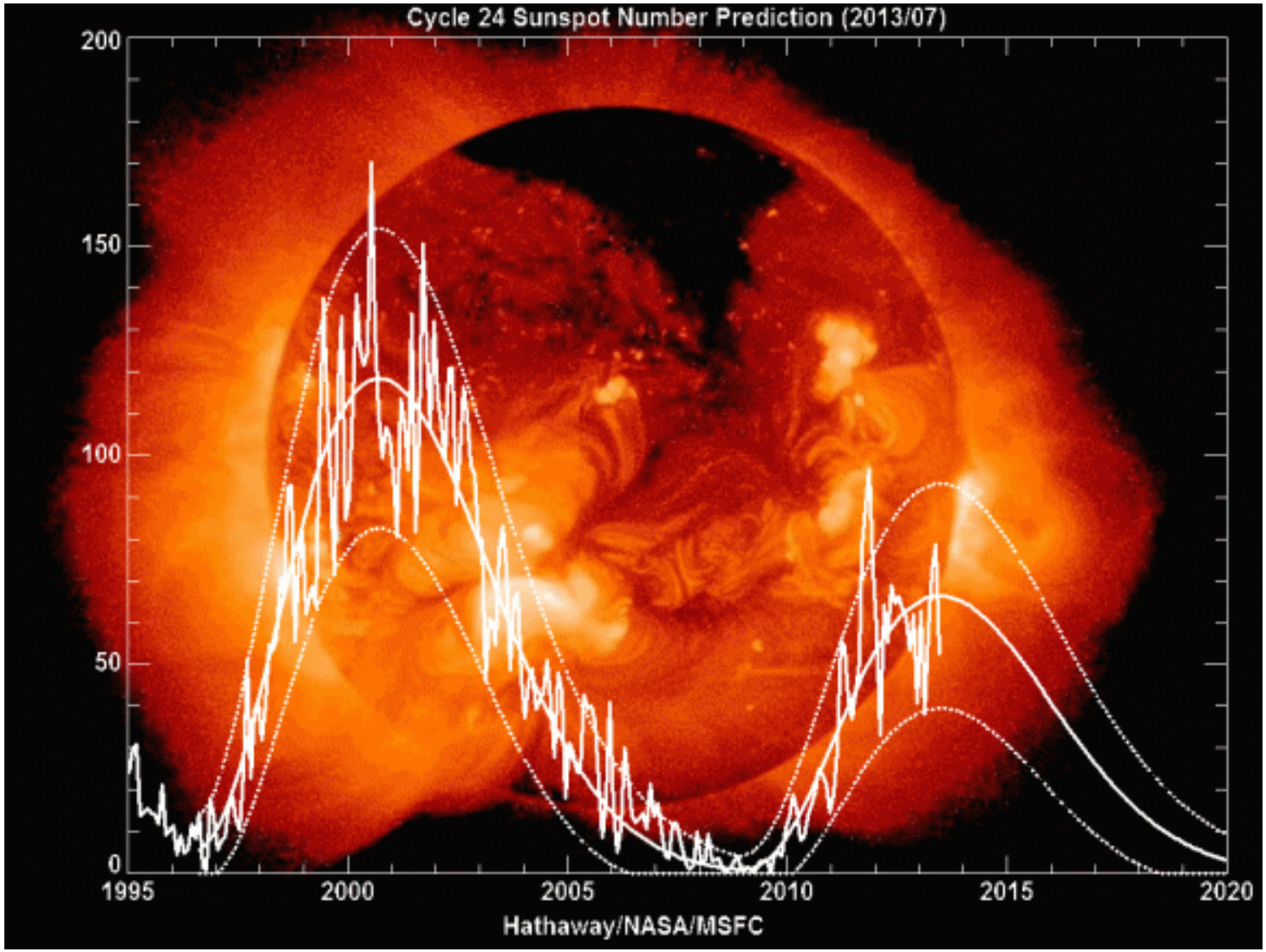


Past Sun Spot Cycles



Sun spots even had a "Great Moderation"

Current cycle (at peak again)



Cute NASA video

- https://www.youtube.com/watch?v=UD5VViT08ME

Sunspots in economics

- **Definition:** a solution is a sunspot solution if it depends on a stochastic variable *from outside system*

- **Model:**

$$\begin{aligned} 0 &= \mathbb{E}H(p_{t+1}, p_t, d_{t+1}, d_t) \\ d_t &: \text{exogenous random variable} \end{aligned}$$

Sunspots in economics (Cass & Shell 1983)

- **Non-sunspot solution:**

$$p_t = f(p_{t-1}, p_{t-2}, \cdots, d_t, d_{t-1}, \cdots)$$

- **Sunspot:**

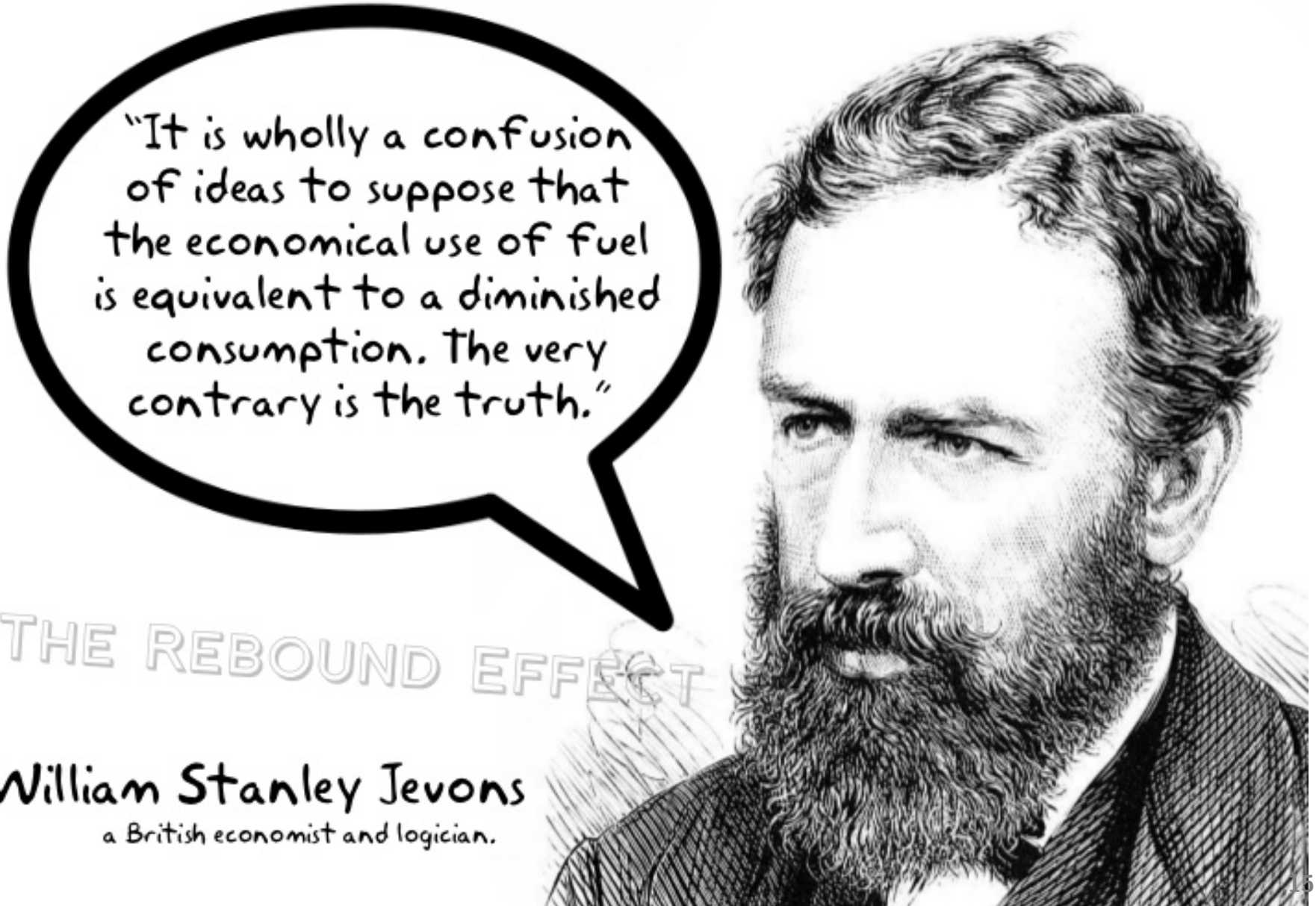
$$\begin{aligned} p_t &= f(p_{t-1}, p_{t-2}, \cdots, d_t, d_{t-1}, \cdots, s_t) \\ s_t &: \text{ random variable with } \mathbb{E}\left[s_{t+1}\right] = 0 \end{aligned}$$

Origin of sunspots in economics

- William Stanley Jevons (1835-82) explored empirical relationship between sunspot activity (that is, the real thing!!!) and the price of corn.

- Fortunately, Jevons had some other contributions as well, such as "Jevons Paradox". His work is considered to be the start of mathematical economics.

Jevons Paradox



Sunspots and science

Why are sunspots attractive

- sunspots: s_t matters, just because agents believe this
  - self-fulfilling expectations don't seem that unreasonable
- sunspots provide many sources of shocks
  - number of sizable fundamental shocks small

Sunspots and science

Why are sunspots not so attractive

- Purpose of science is to come up with predictions
  - If there is one sunspot solution, there are zillion others as well
- Support for the conditions that make them happen not overwhelming
  - you need sufficiently large increasing returns to scale or externality

Overview

1. Getting started
   - simple examples
2. General derivation of Blanchard-Kahn solution
   - When unique solution?
   - When multiple solution?
   - When no (stable) solution?
3. When do sunspots occur?
4. Numerical algorithms and sunspots

Getting started

- **Model:** $y_t = \rho y_{t-1}$

Getting started

- **Model:** $y_t = \rho y_{t-1}$

- infinite number of solutions, independent of the value of ρ

Getting started

- **Model:** $y_{t+1} = \rho y_t$
  y_0 is given

Getting started

- **Model:** $y_{t+1} = \rho y_t$, y_0 is given

- unique solution, independent of the value of ρ

Getting started

- Blanchard-Kahn conditions apply to models that add as a requirement that the series do not explode

Model:

$$y_{t+1} = \rho y_t$$

y_t cannot explode

- $\rho > 1$: unique solution, namely $y_t = 0$ for all t
- $\rho < 1$: many solutions
- $\rho = 1$: many solutions
  - be careful with $\rho = 1$, uncertainty matters

State-space representation

$$Ay_{t+1} + By_t = \varepsilon_{t+1}$$

$$\mathbb{E}\left[\varepsilon_{t+1}|I_t\right] = 0$$

y_t : is an $n \times 1$ vector
$m \leq n$ elements are not determined

some elements of ε_{t+1} are not exogenous shocks but prediction errors

Neoclassical growth model and state space representation

$$\begin{array}{c} \left(\exp(z_t)k_{t-1}^{\alpha} + (1-\delta)k_{t-1} - k_t\right)^{-\gamma} = \\ \mathbb{E}\left[\begin{array}{c} \beta\left(\exp(z_{t+1})k_t^{\alpha} + (1-\delta)k_t - k_{t+1}\right)^{-\gamma} \\ \times\left(\alpha\exp\left(z_{t+1}\right)k_t^{\alpha-1} + 1 - \delta\right) \end{array}\middle| I_t\right] \end{array}$$

or equivalently without $\mathbb{E}\left[\cdot\right]$

$$\begin{array}{c} \left(\exp(z_t)k_{t-1}^{\alpha} + (1-\delta)k_{t-1} - k_t\right)^{-\gamma} = \\ \beta\left(\exp(z_{t+1})k_t^{\alpha} + (1-\delta)k_t - k_{t+1}\right)^{-\gamma} \\ \times\left(\alpha\exp\left(z_{t+1}\right)k_t^{\alpha-1} + 1 - \delta\right) \\ +e_{\mathrm{E},t+1} \end{array}$$

Neoclassical growth model and state space representation

Linearized model:

$$k_{t+1} = a_1 k_t + a_2 k_{t-1} + a_3 z_{t+1} + a_4 z_t + e_{\mathrm{E},t+1}$$

$$z_{t+1} = \rho z_t + e_{z,t+1}$$

$$k_0 \text{ is given}$$

- k_t is end-of-period t capital
  - $\Longrightarrow$ k_t is chosen in t

Neoclassical growth model and state space representation

$$\begin{bmatrix} 1 & 0 & -a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_{t+1} \\ k_t \\ z_{t+1} \end{bmatrix} + \begin{bmatrix} -a_1 & -a_2 & -a_4 \\ -1 & 0 & 0 \\ 0 & 0 & -\rho \end{bmatrix} \begin{bmatrix} k_t \\ k_{t-1} \\ z_t \end{bmatrix} = \begin{bmatrix} e_{\mathrm{E},t+1} \\ 0 \\ e_{z,t+1} \end{bmatrix}$$

Dynamics of the state-space system

$$Ay_{t+1} + By_t = \varepsilon_{t+1}$$

$$\begin{aligned} y_{t+1} &= -A^{-1}By_t + A^{-1}\varepsilon_{t+1} \\ &= Dy_t + A^{-1}\varepsilon_{t+1} \end{aligned}$$

Thus

$$y_{t+1} = D^t y_1 + \sum_{l=1}^{t} D^{t-l} A^{-1} \varepsilon_{l+1}$$

Jordan matrix decomposition

$$D = P\Lambda P^{-1}$$

- Λ is a diagonal matrix with the eigen values of D
- without loss of generality assume that $|\lambda_1| \geq |\lambda_2| \geq \cdots |\lambda_n|$

Let

$$P^{-1} = \begin{bmatrix} \tilde{p}_1 \\ \vdots \\ \tilde{p}_n \end{bmatrix}$$

where $\tilde{p}_i$ is a $(1 \times n)$ vector

Dynamics of the state-space system

$$
\begin{aligned}
y_{t+1} &= D^t y_1 + \sum_{l=1}^{t} D^{t-l} A^{-1} \varepsilon_{l+1} \\
&= P \Lambda^t P^{-1} y_1 + \sum_{l=1}^{t} P \Lambda^{t-l} P^{-1} A^{-1} \varepsilon_{l+1}
\end{aligned}
$$

Dynamics of the state-space system

multiplying dynamic state-space system with P^{-1} gives

$$P^{-1}y_{t+1} = \Lambda^t P^{-1}y_1 + \sum_{l=1}^{t} \Lambda^{t-l} P^{-1} A^{-1} \varepsilon_{l+1}$$

or

$$\tilde{p}_i y_{t+1} = \lambda_i^t \tilde{p}_i y_1 + \sum_{l=1}^{t} \lambda_i^{t-l} \tilde{p}_i A^{-1} \varepsilon_{l+1}$$

recall that y_t is $n \times 1$ and $\tilde{p}_i$ is $1 \times n$. Thus, $\tilde{p}_i y_t$ is a scalar

Model

1. $\tilde{p}_i y_{t+1} = \lambda_i^t \tilde{p}_i y_1 + \sum_{l=1}^{t} \lambda_i^{t-l} \tilde{p}_i A^{-1} \varepsilon_{l+1}$
2. $\mathbb{E}\left[\varepsilon_{t+1} | I_t\right] = 0$
3. m elements of y_1 are not determined
4. y_t cannot explode

Reasons for multiplicity

1. There are free elements in y_1
2. The only constraint on $e_{\mathrm{E},t+1}$ is that it is a prediction error.
   - This leaves lots of freedom

Eigen values and multiplicity

- Suppose that $|\lambda_1| > 1$
- To avoid explosive behavior it *must* be the case that

1. $\tilde{p}_1 y_1 = 0$ and
2. $\tilde{p}_1 A^{-1} \varepsilon_l = 0 \quad \forall l$

How to think about #1?

$$\tilde{p}_1 y_1 = 0$$

- Simply an additional equation to pin down some of the free elements
- Much better: This is the policy rule in the first period

How to think about #1?

$$\tilde{p}_1 y_1 = 0$$

Neoclassical growth model:

- $y_1 = [k_1, k_0, z_1]^T$
- $|\lambda_1| > 1$, $|\lambda_2| < 1$, $\lambda_3 = \rho < 1$
- $\tilde{p}_1 y_1$ pins down k_1 as a function of k_0 and z_1
    - this is the policy function in the first period

How to think about #2?

$$\tilde{p}_1 A^{-1} \varepsilon_l = 0 \quad \forall l$$

- This pins down $e_{\mathrm{E},t}$ as a function of $\varepsilon_{z,t}$
- That is, the prediction error must be a function of the structural shock, $\varepsilon_{z,t}$, and cannot be a function of other shocks,
  - i.e., there are no sunspots

How to think about #2?

$$\tilde{p}_1 A^{-1} \varepsilon_l = 0 \quad \forall l$$

Neoclassical growth model:

- $\tilde{p}_1 A^{-1} \varepsilon_t$ says that the prediction error $e_{\mathsf{E},t}$ of period t is a fixed function of the innovation in period t of the exogenous process, $e_{z,t}$

How to think about #1 combined with #2?

$$\tilde{p}_1 y_t = 0 \quad \forall t$$

- Without sunspots
  - i.e. with $\tilde{p}_1 A^{-1} \varepsilon_t = 0 \quad \forall t$
- k_t is pinned down by k_{t-1} and z_t *in every period*.

Blanchard-Kahn conditions

- Uniqueness: For every free element in y_1, you need one $\lambda_i > 1$
- Multiplicity: Not enough eigenvalues larger than one
- No stable solution: Too many eigenvalues larger than one

How come this is so simple?

- In practice, it is easy to get

$$Ay_{t+1} + By_t = \varepsilon_{t+1}$$

- How about the next step?

$$y_{t+1} = -A^{-1}By_t + A^{-1}\varepsilon_{t+1}$$

- **Bad news**: A is often not invertible
- **Good news**: Same set of results can be derived
  - Schur decomposition (See Klein 2000 and Soderlind 1999)

How to check in Dynare

Use the following command after the model & initial conditions part

```
check;
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Example - x predetermined - 1st order

$$\begin{aligned} x_{t-1} &= \phi x_t + \mathbb{E}_t\left[z_{t+1}\right] \\ z_t &= 0.9 z_{t-1} + \varepsilon_t \end{aligned}$$

- $|\phi| > 1$: Unique stable fixed point
- $|\phi| < 1$: No stable solutions; too many eigenvalues > 1

Example - x predetermined - 1st order

Corresponding state space system:

$$\begin{bmatrix} \phi & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ z_{t+1} \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ z_t \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon_{t+1} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1/\phi & 0 \\ 0 & 0.9 \end{bmatrix}$$

- No sunspots, since $\mathbb{E}_t \left[z_{t+1} \right]$ is the only expectation.
- No multiplicity because of free initial conditions either one starting value for x given.
- So we just need stability $\Longrightarrow |\phi| > 1$

Example - x predetermined - 2nd order

$$\begin{aligned} \phi_2 x_{t-1} &= \mathbb{E}_t \left[\phi_1 x_t + x_{t+1} + z_{t+1} \right] \\ z_t &= 0.9 z_{t-1} + \varepsilon_t \end{aligned}$$

- $\phi_1 = -2.25$, $\phi_2 = -0.5$: Unique stable fixed point
  $(1 + \phi_1 L - \phi_2 L^2) x_t = (1 - 2L)(1 - \frac{1}{4} L) x_t$
- $\phi_1 = -3.5$, $\phi_2 = -3$: No stable solution; too many eigenvalues > 1
  $(1 + \phi_1 L - \phi_2 L^2) x_t = (1 - 2L)(1 - 1.5L) x_t$
- $\phi_1 = -1$, $\phi_2 = -0.25$: Multiple stable solutions; too few eigenvalues > 1
  $(1 + \phi_1 L - \phi_2 L^2) x_t = (1 - 0.5L)(1 - 0.5L) x_t$

Example - x predetermined - 2nd order

Corresponding state space system:

$$\begin{bmatrix} 1 & \phi_1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ x_t \\ z_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & -\phi_2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -0.9 \end{bmatrix} \begin{bmatrix} x_t \\ x_{t-1} \\ z_t \end{bmatrix} = \begin{bmatrix} e_{\mathbb{E},t+} \\ 0 \\ \varepsilon_t \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

- The Λ matrix is for the first numerical example ($\phi_1 = -2.25$, $\phi_2 = -0.5$)

Example - x not predetermined - 1st order

$$\begin{aligned} x_t &= \mathbb{E}_t \left[\phi x_{t+1} + z_{t+1} \right] \\ z_t &= 0.9 z_{t-1} + \varepsilon_t \end{aligned}$$

- $|\phi| < 1$: Unique stable fixed point
- $|\phi| > 1$: Multiple stable solutions; too few eigenvalues > 1

Example - x not predetermined - 1st order

Corresponding state space system:

$$\begin{bmatrix} \phi & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ z_{t+1} \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} x_t \\ z_t \end{bmatrix} = \begin{bmatrix} e_{\mathbb{E},t+1} \\ \varepsilon_{t+1} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \phi & 0 \\ 0 & 0.9 \end{bmatrix}$$

- No sunspots, since $\mathbb{E}_t \left[z_{t+1} \right]$ is the only expectation.
- No multiplicity because of free initial conditions either one starting value for x given.
- So we just need stability $\Longrightarrow |\phi| > 1$

Solutions to linear systems

1. The analysis outlined above
   (requires A to be invertible)
2. Generalized version of analysis above
   (see Klein 2000)

3. Apply time iteration to linearized system
   (I learned this from Pontus Rendahl)

Solutions to linear systems

Model:

$$\Gamma_2 k_{t+1} + \Gamma_1 k_t + \Gamma_0 k_{t-1} = 0$$

or

$$\begin{bmatrix} \Gamma_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_{t+1} \\ k_t \end{bmatrix} + \begin{bmatrix} \Gamma_1 & \Gamma_0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} k_t \\ k_{t-1} \end{bmatrix} = 0$$

Standard approaches

1. The method outlined above
   $\Longrightarrow$ a unique solution of the form

$$k_t = ak_{t-1}$$

   if BK conditions are satisfied

2. Impose that the solution is of the form

$$k_t = ak_{t-1}$$

   and solve for a from

$$\Gamma_2 a^2 k_{t-1} + \Gamma_1 a k_{t-1} + \Gamma_0 k_{t-1} = 0 \quad \forall k_{t-1}$$

Time iteration

- Impose that the solution is of the form

$$k_t = ak_{t-1}$$

- Use time iteration scheme, starting with $a_{[1]}$
- Recall that time iteration means using the guess for *tomorrows* behavior and then solve for *todays* behavior

(This simple procedure was pointed out to me by Pontus Rendahl)

Time iteration

- Follow the following iteration scheme, starting with $a_{[1]}$
  - Use $a_{[i]}$ to describe next period's behavior. That is,

$$\Gamma_2 a_{[i]} k_t + \Gamma_1 k_t + \Gamma_0 k_{t-1} = 0$$

(note the difference with last approach on previous slide)

  - Obtain $a_{[i+1]}$ from

$$(\Gamma_2 a_{[i]} + \Gamma_1) k_t + \Gamma_0 k_{t-1} = 0$$

$$k_t = -\left(\Gamma_2 a_{[i]} + \Gamma_1\right)^{-1} \Gamma_0 k_{t-1}$$

$$a_{[i+1]} = -\left(\Gamma_2 a_{[i]} + \Gamma_1\right)^{-1} \Gamma_0$$

Advantages of time iteration

- It is simple even if the "A matrix" is not invertible.
  (the inversion required by time iteration seems less problematic in practice)

- Since time iteration is linked to value function iteration, it has nice convergence properties

Example

$$k_{t+1} - 2k_t + 0.75k_{t-1} = 0$$

- The two solutions are

$$k_t = 0.5k_{t-1} \ \& \ k_t = 1.5k_{t-1}$$

- Time iteration on $k_t = a_{[i]}k_{t-1}$ converges to stable solution for all initial values of $a_{[i]}$ except 1.5.

References and Acknowledgements

- Larry Christiano taught me (a long time ago) this simple way of deriving the BK conditions and I think that I did not even change the notation.
- Blanchard, Olivier and Charles M. Kahn, 1980,The Solution of Linear Difference Models under Rational Expectations, Econometrica, 1305-1313.
- Den Haan, Wouter J., 2007, Shocks and the Unavoidable Road to Higher Taxes and Higher Unemployment, Review of Economic Dynamics, 348-366.
    - simple model in which the size of the shocks has long-term consequences
- Farmer, Roger, 1993, The Macroeconomics of Self-Fulfilling Prophecies, The MIT Press.
    - textbook by the pioneer
- Klein, Paul, 2000, Using the Generalized Schur form to Solve a Multivariate Linear Rational Expectations Model, Journal of Economic Dynamics and Control, 1405-1423.
    - in case you want to do the analysis without the simplifying assumption that A is invertible
- Soderlind, Paul, 1999, Solution and estimation of RE macromodels with optimal policy, European Economic Review, 813-823
    - also doesn't assume that A is invertible; possibly a more accessible paper

Parameterized Expectations Algorithm

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Overview

- Two PEA algorithms
- Explaining stochastic simulations PEA
- Advantages and disadvantages
- Improvements of Maliar, Maliar & Judd
- Extensions
  - learning
  - combining with perturbation

Model

$$
\begin{aligned}
c_t^{-\nu} &= \mathsf{E}_t\left[\beta c_{t+1}^{-\nu}\left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta\right)\right] \\
c_t + k_{t+1} &= z_t k_t^{\alpha} + (1-\delta)\, k_t \\
\ln(z_{t+1}) &= \rho \ln(z_t) + \varepsilon_{t+1} \\
&\quad \varepsilon_{t+1} \sim N(0, \sigma^2) \\
&\quad k_1, z_1 \text{ given}
\end{aligned}
$$

k_t is beginning-of-period t capital stock

Two types of PEA

1. Standard projections algorithm:
   1. parameterize $\mathrm{E}_t[\cdot]$ with $P_n(k_t, z_t; \eta_n)$
   2. solve c_t from
   $$c_t = (P_n(k_t, z_t; \eta_n))^{-1/\nu}$$
   and k_{t+1} from budget constraint
2. Stochastic (simulations) PEA

Stochastic PEA based on simulations

1. Simulate $\{z_t\}_{t=1}^{T}$
2. Let η_n^1 be initial guess for η_n

Stochastic PEA

3. Iterate until η_n^i converges using following scheme
   1. Generate $\{c_t, k_{t+1}\}_{t=1}^{T}$ using

$$\begin{aligned} c_t^{-\nu} &= P_n(k_t, z_t; \eta_n^i) \\ k_{t+1} &= z_t k_t^{\alpha} + (1-\delta) k_t - c_t \end{aligned}$$

   2. Generate $\{y_{t+1}\}_{t=1}^{T-1}$ using

$$y_{t+1} = \beta c_{t+1}^{-\nu} \left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right)$$

   3. Let

$$\hat{\eta}_n^i = \arg\min_{\eta} \sum_{t=T_{\text{begin}}}^{T} \frac{\left(y_{t+1} - P_n(k_t, z_t; \eta)\right)^2}{T}$$

   4. Update using

$$\eta_n^{i+1} = \omega \hat{\eta}_n^i + (1-\omega) \eta_n^i \text{ with } 0 < \omega \leq 1$$

Stochastic PEA

- $T_{\text{begin}} >> 1$ (say 500 or 1,000)
  - ensures possible bad period 1 values don't matter
- $\omega < 1$ improves stability
  - ω is called "dampening" parameter

Stochastic PEA

- Idea of regression:

$$y_{t+1} \approx P_n(k_t, z_t; \eta) + u_{t+1},$$

- u_{t+1} is a prediction error $\Longrightarrow$ u_{t+1} is orthogonal to regressors
- Suppose

$$P_n(k_t, z_t; \eta) = \exp\left(a_0 + a_1 \ln k_t + a_2 \ln z_t\right).$$

- You are *not* allowed to run the linear regression

$$\ln y_{t+1} = a_0 + a_1 \ln k_t + a_2 \ln z_t + \tilde{u}_{t+1}$$

  Why not?

PEA & RE

- Suppose η_n^* is the fixed point we are looking for
- So with η_n^* we get best predictor of y_{t+1}
- Does this mean that solution is a rational expectations equilibrium?

Disadvantages of stoch. sim. PEA

- The inverse of $X'X$ may be hard to calculate for higher-order approximations
- Regression points are clustered $\Longrightarrow$ low precission
  - recall that even equidistant nodes is not enough for uniform convergence
    "nodes" are even less spread out with stochastic PEA)

Disadvantages of stochastic PEA

- Projection step has sampling error
  - this disappears at slow rate (especially with serial correlation)

Advantages of stoch. sim. PEA

- Regression points are clustered

  $\Longrightarrow$ better fit *where it matters* **IF** functional form is poor

  (with good functional form it is better to spread out points)

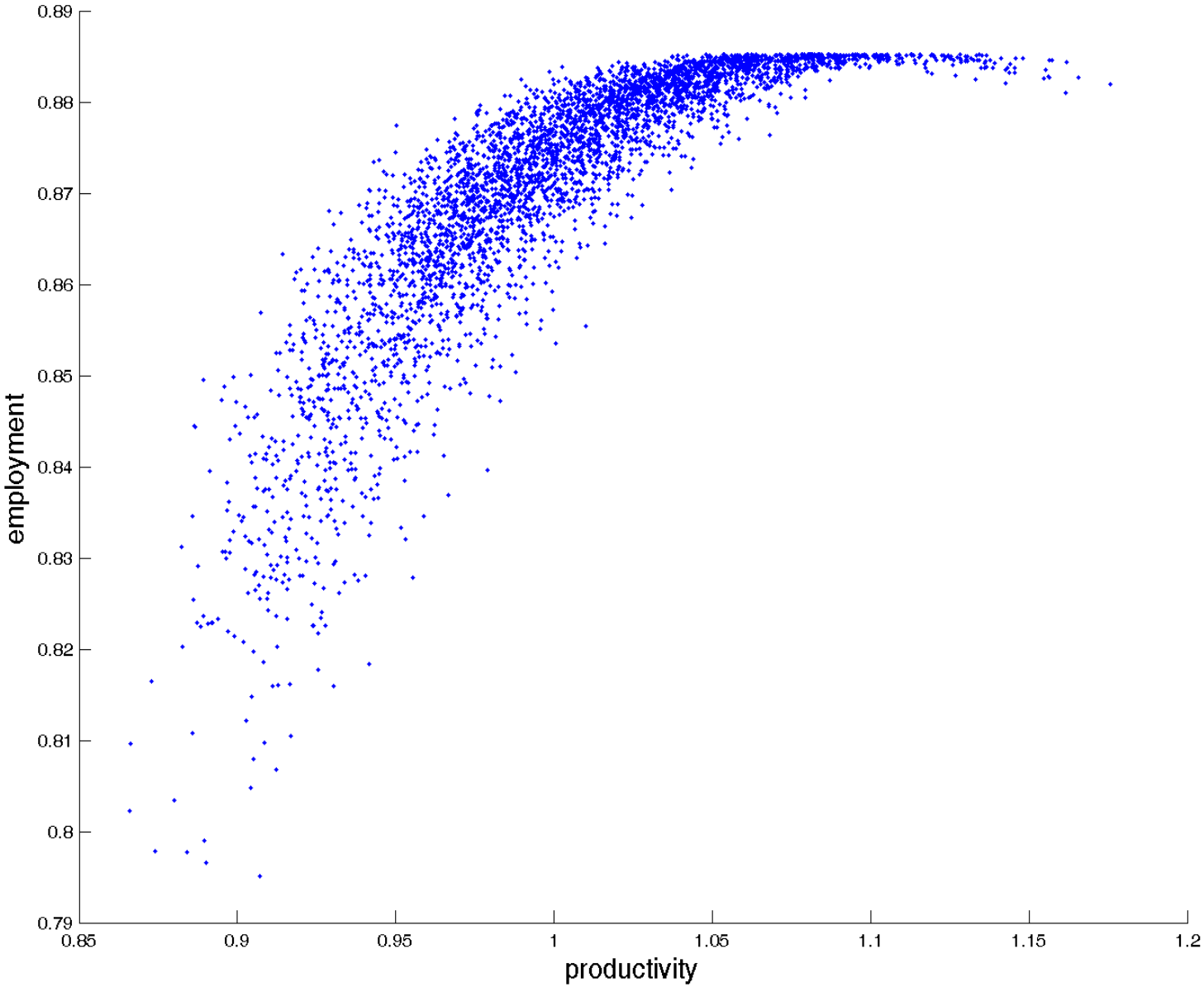
Advantages of stoch. sim. PEA

- Grid: you may include impossible points

  Simulation: model iself tells you which nodes to include

  - (approximation also important and away from fixed point you may still get in weird places of the state space)

Odd shapes ergodic set in matching model



Improvements proposed by Maliar, Maliar & Judd

1. Use flexibility given to you
2. Use $\widehat{\mathsf{E}}[y_{t+1}]$ instead of y_{t+1} as regressand
   - $\widehat{\mathsf{E}}[y_{t+1}]$ is numerical approximation of $\mathsf{E}[y_{t+1}]$
   - even with poor approximation the results improve !!!
3. Improve regression step

Use flexibility

Many E[]'s to approximate.

1. Standard approach:

$$c_t^{-\nu} = \mathsf{E}_t \left[\beta c_{t+1}^{-v} \alpha \beta c_{t+1}^{-\nu} \left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right) \right]$$

2. Alternative:

$$k_{t+1} = \mathsf{E}_t \left[k_{t+1} \beta \alpha \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\nu} \left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right) \right]$$

- Such transformations can make computations easier, *but* can also affect stability of algorithm (for better or worse)

E[y] instead of y as regressor

- $\mathsf{E}[y_{t+1}] = \mathsf{E}[f(\varepsilon_{t+1})]$ with $\varepsilon_{t+1} \sim N(0, \sigma^2)$

  $\Longrightarrow$ Hermite Gaussian quadrature can be used

  (MMJ: using $\widehat{\mathsf{E}}[y_{t+1}]$ calculated using one node is better than using y_{t+1})

- Key thing to remember: sampling uncertainty is hard to get rid off

E[y] instead of y as regressor

- Suppose:

$$\begin{aligned} y_{t+1} &= \exp\left(a_o + a_1 \ln k_t + a_2 \ln z_t\right) + u_{t+1} \\ u_{t+1} &= \text{prediction error} \end{aligned}$$

- Then you **cannot** estimate coefficients using LS based on

$$\ln\left(y_{t+1}\right) = a_o + a_1 \ln k_t + a_2 \ln z_t + u^*_{t+1}$$

- You have to use non-linear least squares

E[y] instead of y as regressor

- Suppose:

$$\begin{aligned} \mathrm{E}\left[y_{t+1}\right] &= \exp\left(a_o + a_1 \ln k_t + a_2 \ln z_t\right) + \bar{u}_{t+1} \\ \bar{u}_{t+1} &= \text{numerical error} \end{aligned}$$

- Then you **can** estimate coefficients using LS based on

$$\ln \mathrm{E}\left[y_{t+1}\right] = a_o + a_1 \ln k_t + a_2 \ln z_t + \bar{u}^*_{t+1}$$

- Big practical advantage

Simple ways to improve regression

1. Hermite polynomials and scaling
2. LS-Singular Value Decomposition
3. Principal components

Simple ways to improve regression

- The main underlying problem is that $X'X$ is ill conditioned which makes it difficult to calculate $X'X$
- This problem is reduced by

1. Scaling so that each variable has zero mean and unit variance
2. Hermite polynomials

Hermite polynomials; Definition

$$P_n(x) = \sum_{j=0}^{n} a_j H_j(x)$$

where the basis functions, $H_j(x)$, satisfy

$$\mathrm{E}\left[H_i(x) H_j(x)\right] = 0 \text{ for } i \neq j$$

$$\text{if } x \sim N(0,1)$$

Hermite polynomials; Construction

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= x \\ H_{m+1}(x) &= xH_m(x) - mH_{m-1}(x) \text{ for } j > 1 \end{aligned}$$

This gives

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= x \\ H_2(x) &= x^2 - 1 \\ H_3(x) &= x^3 - 3x \\ H_4(x) &= x^4 - 6x^2 + 3 \\ H_5(x) &= x^5 - 10x_3 + 15x \end{aligned}$$

One tricky aspect about scaling

Suppose one of the explanatory variables is

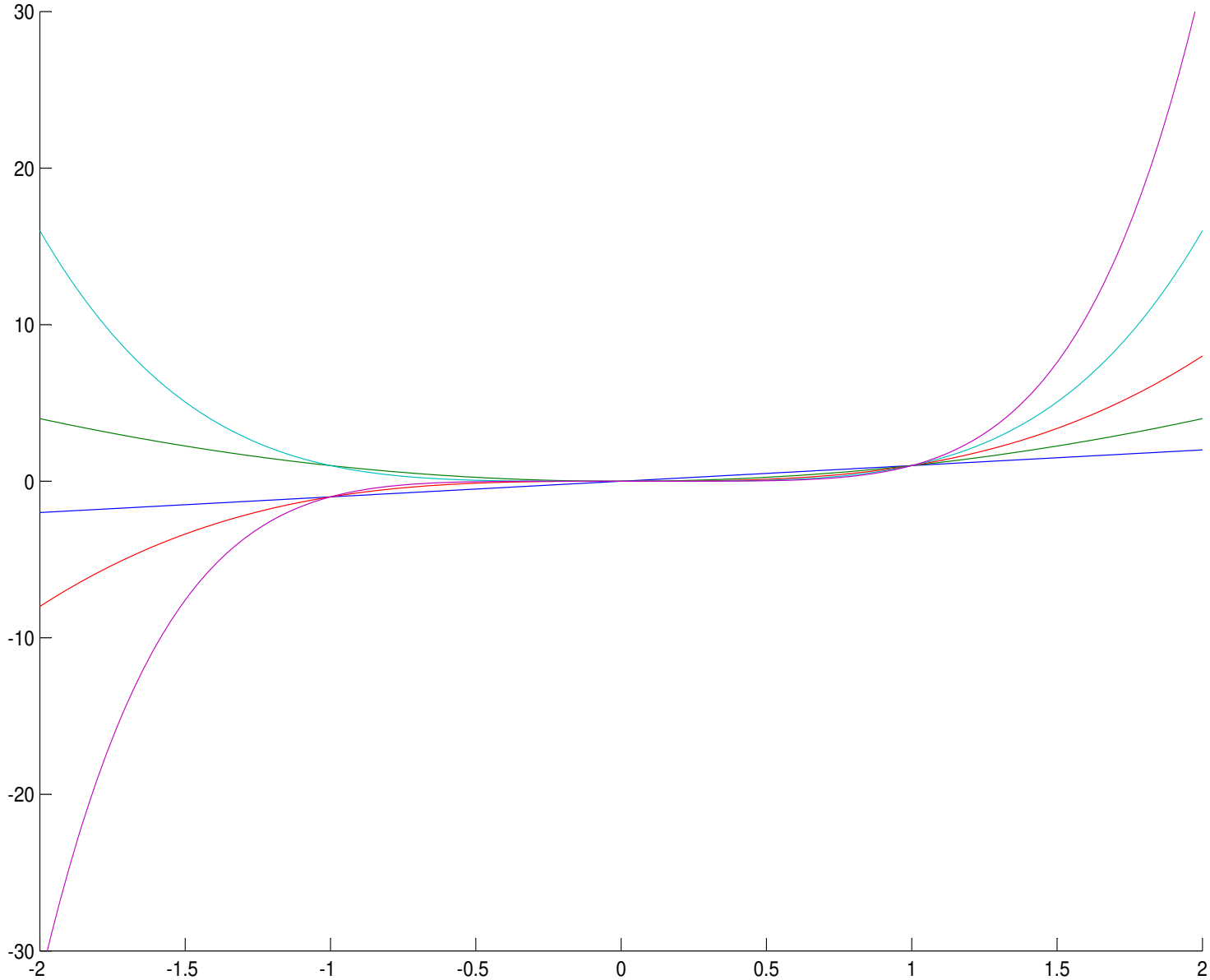
$$x_t = \frac{k_t - M_T}{S_T}$$

$$M_T = \sum_{t=1}^{T} k_t/T \;\&\; S_T = \left(\sum_{t=1}^{T} (k_t - M(k_t)^2 \right)/T)^{1/2}$$

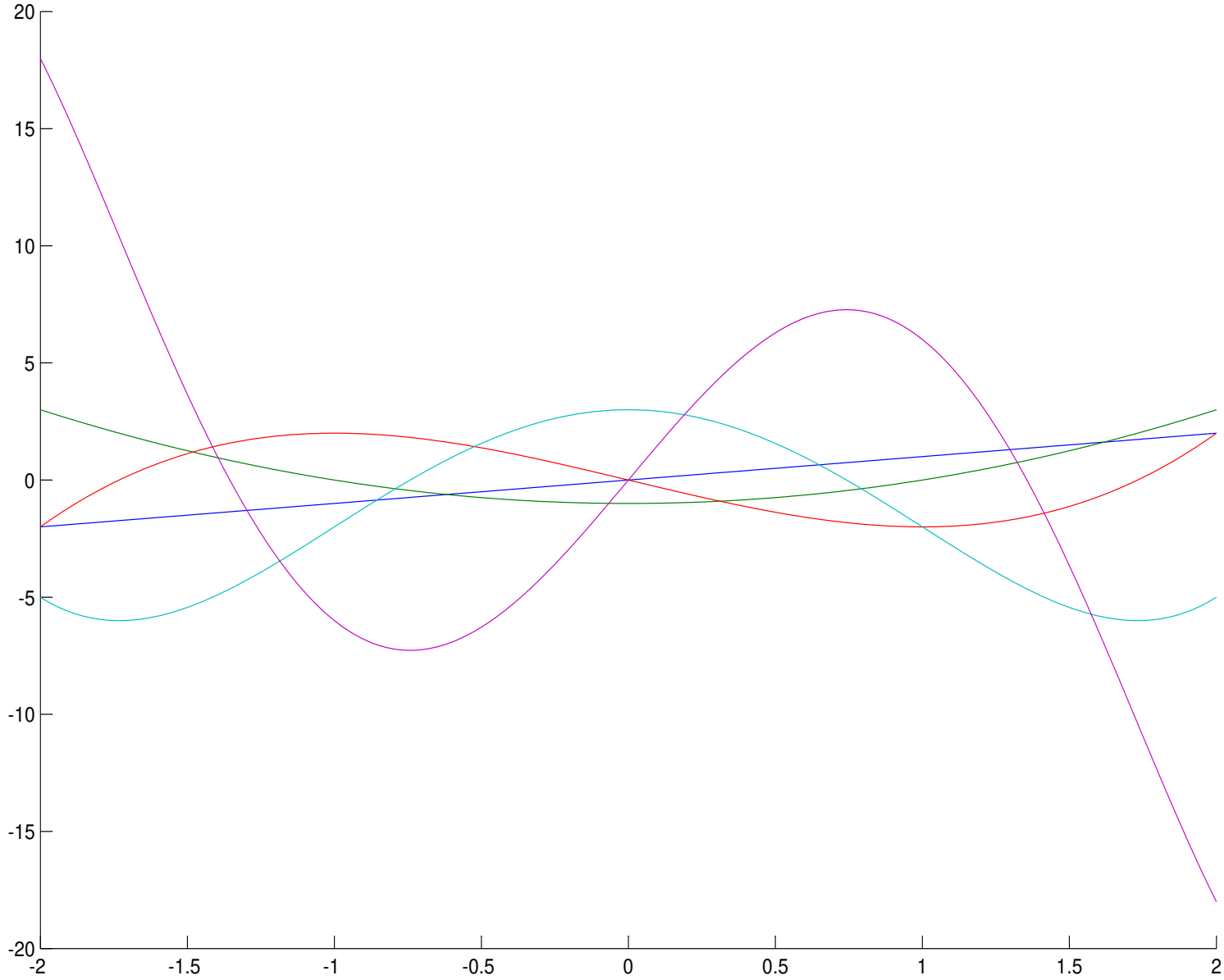
One tricky aspect about scaling

- $\Longrightarrow$ each iteration the explanatory variables change (since M and S change)
- $\Longrightarrow$ taking a weighted average of old and new coefficient is odd
- I found that convergence properties can be quite bad
- In principle you can avoid problem by rewriting polynomial, but that is tedious for higher-order
- So better to keep M_T and S_T fixed across iterations

Two graphs say it all; regular polynomials



Two graphs say it all; Hermite polynomials



LS-Singular Values Decomposition

$$\widehat{\beta} = \left(X'X\right)^{-1} X'Y = VS^{-1}U'Y$$

- Goal: avoid calculating $X'X$ explicitly
- SVD of the $(T \times n)$ matrix X :

$$\begin{array}{rcl} X & = & USV' \\ U & : & (T \times n) \text{ orthogonal matrix} \\ S & : & (n \times n) \text{ diagonal matrix with singular values } s_1 \geq s_2 \geq \cdots \\ V & : & (n \times n) \text{ orthogonal matrix} \end{array}$$

- s_i is the sqrt of i^{th} eigen value

LS-Singular Values Decomposition

In Matlab

```
[U,S,V]=svd(X,0);
```

Principal components

- With many explanatory variables use principle components
  - SVD: $X = USV'$ where X is demeaned
  - Principle components: $Z = XV$
  - Properties Z_i : mean zero and variance s_i^2
- Idea: exclude principle components corresponding to lower eigenvalues
- But check with how much R^2 drops

PEA and learning

- Traditional algorithm:
  - simulate an economy using belief η_n^i
  - formulate new belief η_n^{i+1}
  - simulate *same* economy using belief η_n^{i+1}

PEA and learning

- Alternative algorithm to find *fixed point*
  - simulate T observations using belief η_n^{T-1}
  - formulate new belief η_n^T
  - generate 1 more observation
  - use $T+1$ observations to formulate new belief η^{T+1}
  - continue
- Convergence properties can be problematic

PEA and learning

- Modification of alternative algorithm is economically interesting
  - simulate T observations using belief η_n^{T-1}
  - use τ observations to formulate new belief η_n^T
  - generate 1 more observation
  - use last τ observations to formulate new belief η^{T+1}
  - continue
- Beliefs are based on limited past $\Longrightarrow$ time-varying beliefs

PEA and learning

- Suppose the model has different regimes
  - e.g. high productivity and low productivity regime
  - agents do not observe regime$\Longrightarrow$ it makes sense to use limited number of past observations
- With the above algorithm agents gradually learn new law of motion

PEA and perturbation

- True in many macroeconomic models:
  - perturbation generates accurate solution of "real side" of the economy
  - perturbation does not generates accurate solution of asset prices
  - real side does not at all or not much depend on asset prices
- Then solve for real economy using perturbation and for asset prices using PEA
  - one-step algorithm (no iteration needed)

References

- Den Haan, W.J. and A. Marcet, 1990, Solving the stochastic growth model with parameterized expectations, Journal of Business and Economic Statistics.
- Den Haan, W.J., Parameterized expectations, lecture notes.
- Heer, B., and A. Maussner, 2009, Dynamic General Equilibrium Modeling.
- Judd, K. L. Maliar, and S. Maliar, 2011, One-node quadrature beats Monte Carlo: A generlized stochastic simulation algorithm, NBER WP 16708
- Judd, K. L. Maliar, and S. Maliar, 2010, Numerically stable stochastic methods for solving dynamics models, NBER WP 15296

Value Function Iteration versus Euler equation methods

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Overview

1. How to do value function iteration (VFI)
2. VFI versus Euler equation methods
    1. convergence
    2. speed
    3. complex problems

Bellman equation

$$V(x) = \max_{x_{+1} \in \Gamma(x)} U(x, x_{+1}) + \mathsf{E}_t \left[\beta V(x_{+1}) \right]$$

Essence of VFI

- $V^i(x)$: flexible functional form
  - piecewise linear (or higher-order spline)
  - discrete valued function (if $\Gamma(x)$ has $\chi < \infty$ elements)
  - quadratic (or higher-order polynomial)

- $V^{i+1}(x)$ is obtained from

$$V^{i+1}(x) = \max_{x_{+1} \in \Gamma(x)} U(x, x_{+1}) + E_t \left[\beta V^i(x_{+1}) \right]$$

Essence of VFI

- This works in general
- However, on a computer the functional form of $V^i(x)$ must stay the same
  (so computer can store coefficients characterizing function)

Possible ways to implement VFI

1. Linear-Quadratic
   - $U(\cdot)$ is quadratic and constraints are linear
     $\Longrightarrow V^i(\cdot)$ would remain quadratic
   - !!! To get a true first-order approximation to policy function you cannot take linear approximation of constraints
     $\Longrightarrow$ either get rid of constraint by substitution or use the "correct" LQ approximation (see perturbation slides)

2. Discrete grid $\Longrightarrow \Gamma(x)$ and $V(x)$ have finite # of elements

Possible ways to implement VFI

3. Piecewise linear

- *choices* are no longer constrained to be on grid
- $V^i(\cdot)$ is characterized by function values on grid
- Simply do maximization on grid

4. Regular polynomial

- *choices* are no longer constrained to be on grid
- calculate values V on grid
- obtain V^{i+1} by fitting polynomial through calculated point

Convergence

- There are several convergence results for VFI
- Some such results for Euler equation methods
  - but you have to do it right (e.g. use time & not fixed-point iteration)
- But especially for more complex problems, VFI is more likely to converge

Speed; algorithm choice

- VFI: because of the max operator you typically can only iterate
  - slow if discount factor is close to 1
- Euler equation method have more options
  - calculating fixed point directly with equation solver typically faster

Speed; impact choices on V & Euler

VFI tends to be slow in many typical economic applications

- Reason: value function is flat $\Longrightarrow$ hard to find max
  - important to be aware of this
  - Krusell and Smith (1996) show that utility loss of keeping capital stock constant is minor in neoclassical growth model
  - But shouldn't a flat utility function be problematic for Euler eq. methods as well?

Speed; impact choices on V & Euler

Example to show Euler eq. methods less affected by flatness

$$\max_{x_1,x_2} x_1^{1-\nu} + x_2^{1-\nu}$$

$$\text{s.t. } x_1 + x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Speed; impact choices on V & Euler

Consider a *huge* move away from optimum

ν	$u(1,1)$	$u(2,0)$	consumption equivalent loss
0.01	2	1.9862	0.7%
0.001	2	1.9986	0.07%

Speed; impact choices on V & Euler

First-order condition:

$$\left(\frac{x_1}{x_2}\right)^{-\nu} = 1 \text{ or } x_1 = 1^{-1/\nu} \times x_2$$

Marginal rates of substitution:

ν	$x_1 = x_2 = 1$	$x_1 = 2, x_2 = 0$
0.01	1	∞
0.001	1	∞

Dealing with complex problems

- Both VFI and Euler-equation methods can deal with inequality constraints
- Euler equations require first-order conditions to be sufficient
  - this requires concavity (utility function) and convex opportunity set
  - this is not always satisfied

Non-convex problem - example

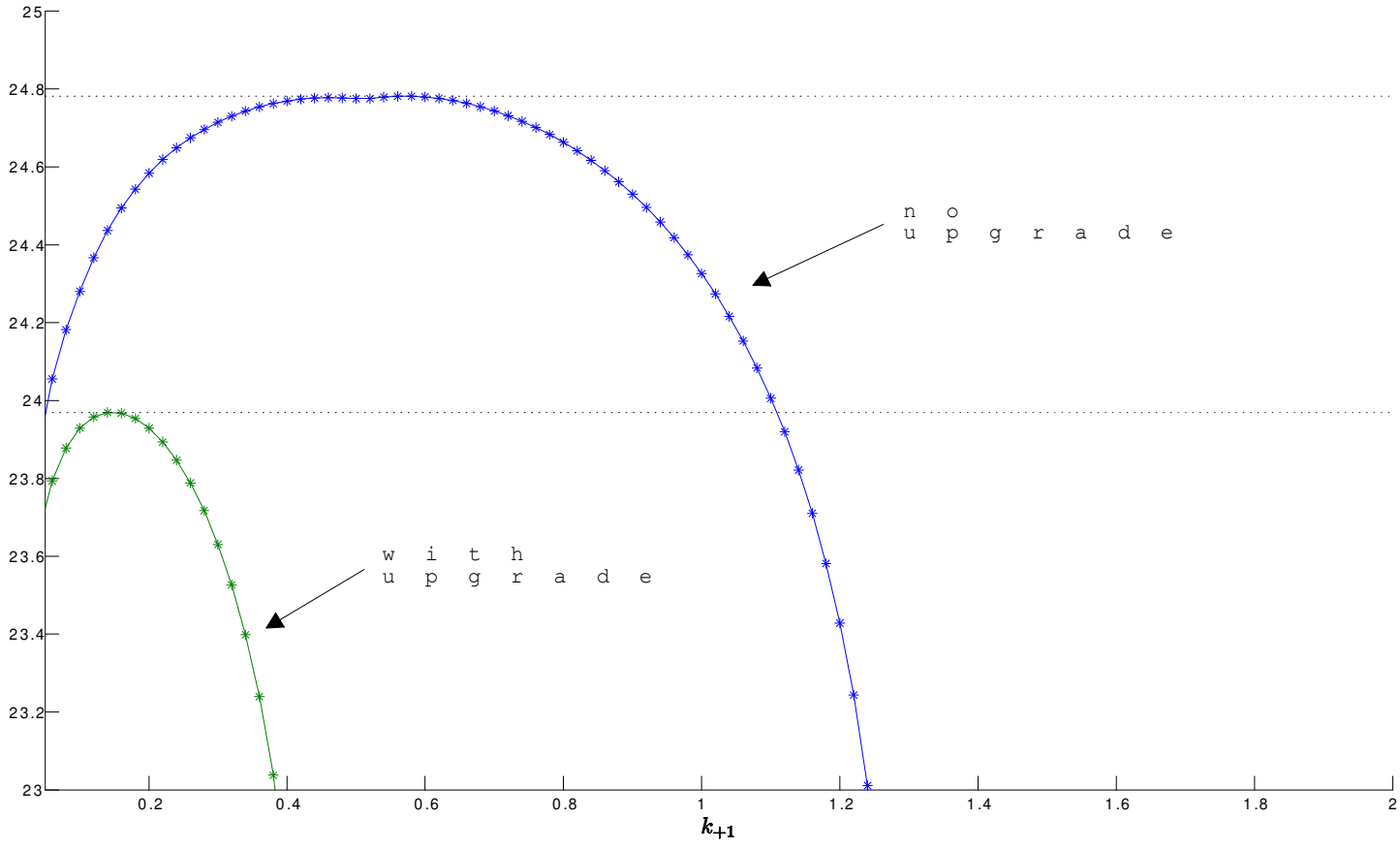
Environment:

- Two technologies:
  - $y_t = k_t^\alpha$
  - $y_t = Ak_t^\alpha$ with $A > 1$
- Higher-productivity technology can be used after paying a one-time cost ψ

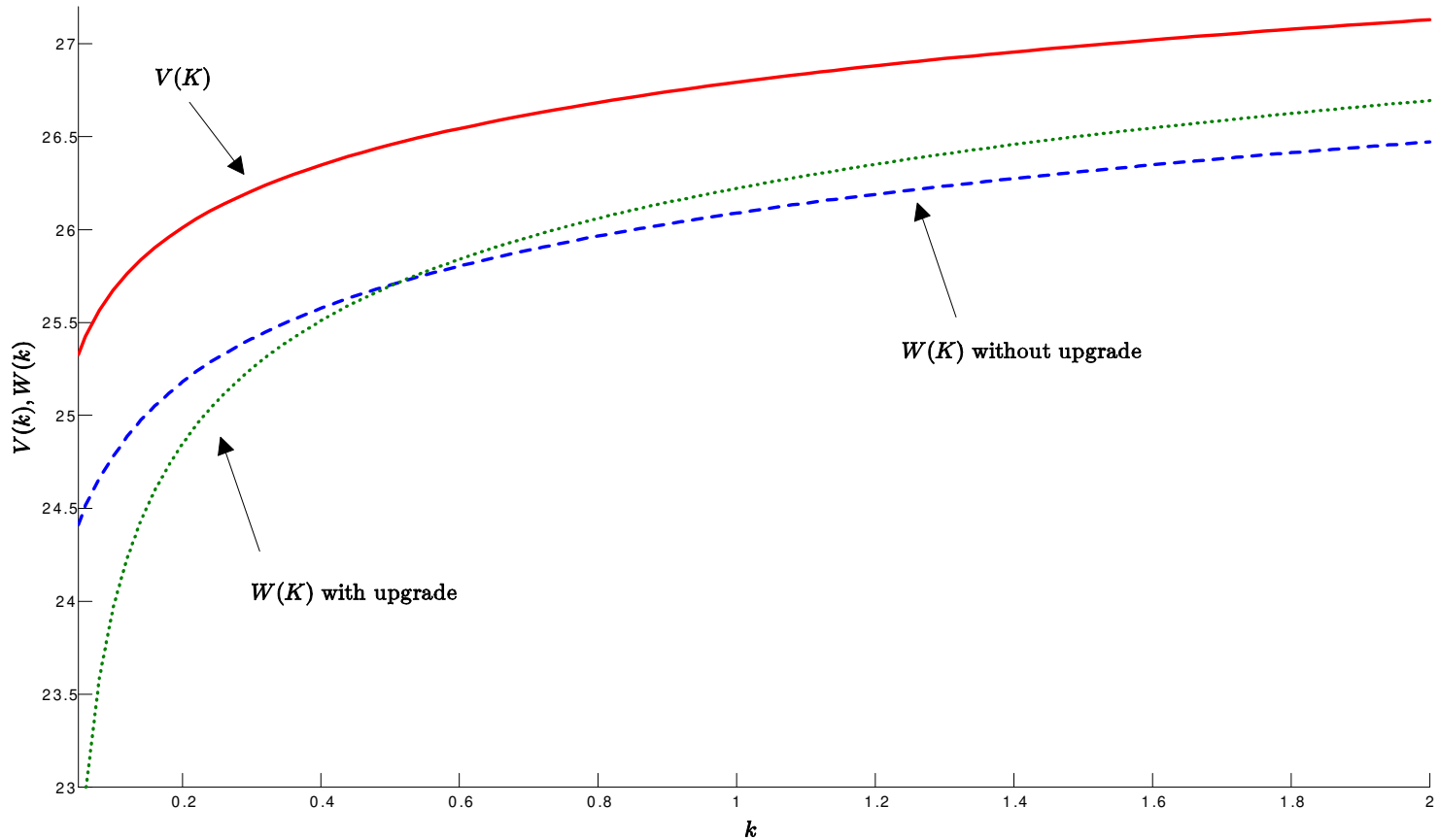
Non-convex problem - example

$$
\begin{aligned}
W(k) &= \max \left\{ \begin{array}{c} \max_{k_{+1}} k^{\alpha} - k_{+1} + \beta W\left(k_{+1}\right), \\ \max_{k_{+1}} k^{\alpha} - k_{+1} - \psi + \beta V\left(k_{+1}\right) \end{array} \right\} \\
V(k) &= \max_{k_{+1}} A k^{\alpha} - k_{+1} + \beta V\left(k_{+1}\right)
\end{aligned}
$$

RHS Bellman equation for low capital stock (k=0.1)



Ultimate value function



References

- Slides on perturbation; available online.
- Slides on projection methods; available online.
- Judd, K. L., 1998, Numerical Methods in Economics.
- Krusell, P. & A. Smith, 1996. Rules of thumb in macroeconomic equilibrium A quantitative analysis, Journal of Economic Dynamics and Control.
- Rendahl, P., 2006, Inequality constraints in recursive economies.
    - shows that time-iteration converges even in the presence of inequality constraints

Accuracy tests

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How to check for accuracy

1. Informal accuracy tests
2. Formal accuracy tests

Informal accuracy tests

These are possibly more important than formal ones

1. "Play" with your model/algorithm
    1. Understand properties of the model
    2. Change parameter values and understand how model properties change
    3. Open up the black box

Informal accuracy tests

2. Solve your model in a different way
    1. Linear instead of log-linear
    2. Use model equations to substitute out variables
    3. Approximate something else
        - c_t instead of k_{t+1}
        - $c^{-\gamma}$ instead of c_t

Formal accuracy tests

1. Euler-equation errors
   - require numerical integration
     (but this is not that difficult to do)
2. Dynamic Euler-equation errors
   - also requires numerical integration
3. Welfare measures (be careful)
4. DenHaan-Marcet (DHM) accuracy test
   - simple, but hard to interpret

Idea behind most accuracy tests

Model:

$$\mathsf{E}\left[f(x_{t-1}, x_t, y_t, y_{t+1})|I_t\right] = 0$$

where $\mathsf{E}[f(\cdot)|I_t]$ is the Euler-equation errror

Accuracy tests:

- Euler-equation error: $\mathsf{E}[f(\cdot)|I_t]$ should be zero at *many* points in state space

Euler-eq. error & standard growth model

$$f_t = -c_t^{-\gamma} + \beta c_{t+1}^{-\gamma}(\alpha \exp(z_{t+1}) k_t^{\alpha-1} + 1 - \delta)$$

with

$$z_{t+1} = \rho z_t + \sigma e_{t+1}$$

Euler-equation errors

- True solution satisfies

$$\mathsf{E}\left[f\left(x_{t-1}, x_t, y_t, y_{t+1}\right) | I_t\right] = 0$$

  for *all* points in the state space

- This can be checked for *any* numerical solution (including perturbation solutions) at *many* points in the state space

How to deal with integration?

- Easy if shocks have discrete support
- Numerical integration
  (this must be done accurately)

Growth model with discrete innovations

$$\max_{\{c_t,k_t\}_{t=1}^{\infty}} \mathsf{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma}-1}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_t = \exp(z_t) k_{t-1}^{\alpha} + (1-\delta)k_t \tag{1}$$

$$z_t = \rho z_{t-1} + \sigma e_t, \tag{2}$$

$$e_t = \begin{cases} +1 \text{ with probability } 1/2 \\ \text{-1 with probability } 1/2 \end{cases}$$

Basic idea

1. Construct *fine* grid with values for k_{-1} and z
2. Euler-equation error at (k_{-1}, z) equals

$$
\begin{aligned}
&-\boldsymbol{c}(k_{-1}, z)^{-\gamma} \\
&+0.5 * \beta \boldsymbol{c}(\boldsymbol{k}, \rho z + \sigma)^{-\gamma} (\alpha \exp(\rho z + \sigma) \boldsymbol{k}^{\alpha - 1} + 1 - \delta) \\
&+0.5 * \beta \boldsymbol{c}(\boldsymbol{k}, \rho z - \sigma)^{-\gamma} (\alpha \exp(\rho z - \sigma) \boldsymbol{k}^{\alpha - 1} + 1 - \delta)
\end{aligned}
$$

with $\boldsymbol{k} = \boldsymbol{k}(k_{-1}, z)$

When is a solution accurate

- When Euler eq. errors are small at many points
- Problem: magnitude of errors is hard to interpret

Interpretable Euler-equation errors

- At each grid point calculate *two* consumption values
  1. $\boldsymbol{c}(k_{-1}, z)$ using the numerical approximation
  2. implied value, $\boldsymbol{c}_{imp}(k_{-1}, z)$, using

$$\boldsymbol{c}_{imp}(k_{-1}, z) = g^{-1/\gamma}$$

with

$$g = \begin{array}{l} +0.5 * \beta \boldsymbol{c}(\boldsymbol{k}, z, +\sigma)^{-\gamma}(\alpha \exp(\rho z + \sigma)\boldsymbol{k}^{\alpha-1} + 1 - \delta) \\ +0.5 * \beta \boldsymbol{c}(\boldsymbol{k}, z, -\sigma)^{-\gamma}(\alpha \exp(\rho z - \sigma)\boldsymbol{k}^{\alpha-1} + 1 - \delta) \end{array}$$

that is, value implied by accurately calculated RHS of Euler equation

Interpretable Euler-equation errors

- Euler-equation error is equal to

$$\left| \frac{\boldsymbol{c}(k_{-1}, z) - \boldsymbol{c}_{imp}(k_{-1}, z)}{\boldsymbol{c}_{imp}(k_{-1}, z)} \right|$$

What to do with the errors?

- Calculate maximum and average of the errors
- Investigate
  - Pattern (e.g., are errors always of the same sign)
  - Are nodes with largest errors very likely?
  - What happens at nodes with largest errors?
    For example, if consumption is very small at those nodes, then small basically irrelevant errors may show up as large percentage errors

Growth model with continuous support

$$\max_{\{c_t,k_t\}_{t=1}^{\infty}} \mathsf{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma}-1}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_t = \exp(z_t) k_{t-1}^{\alpha} + (1-\delta) k_t$$

$$z_t = \rho z_{t-1} + \sigma e_t,$$

$$e_t \sim N(0,1)$$

Calculate conditional expectation

- Given are k_{-1}, z, and policy function $g(k_{-1}, z)$
- $\delta = 1$ to simplify notation

Calculate conditional expectation

Use $k = g(k_{-1}, z)$ to get

$$
\begin{aligned}
& \mathsf{E}\left[\frac{\beta \exp(z_{+1})\alpha k^{\alpha-1}}{c_{t+1}}\right] \\
= \quad & \mathsf{E}\left[\frac{\beta \exp(z_{+1})\alpha k^{\alpha-1}}{\exp(z_{+1})k^{\alpha} - k_{+1}}\right] \\
= \quad & \mathsf{E}\left[\frac{\beta \exp(z_{+1})\alpha k^{\alpha-1}}{\exp(z_{+1})k^{\alpha} - g(k, z_{+1})}\right] \\
= \quad & \mathsf{E}\left[\frac{\beta \exp(\rho z + \sigma\varepsilon_{+1})\alpha k^{\alpha-1}}{\exp(\rho z + \sigma\varepsilon_{+1})k^{\alpha} - g(k, \rho z + \sigma\varepsilon_{+1})}\right]
\end{aligned}
$$

Conditional expectation

$$\mathsf{E}\left[\frac{\beta\exp(\rho z+\sigma\varepsilon_{+1})\alpha k^{\alpha-1}}{\exp(\rho z+\sigma\varepsilon_{+1})k^{\alpha}-g(k,\rho z+\sigma\varepsilon_{+1})}\right]$$

$$=\int_{-\infty}^{\infty}\frac{\beta\exp(\rho z+\sigma\varepsilon_{+1})\alpha k^{\alpha-1}}{\exp(\rho z+\sigma\varepsilon_{+1})k^{\alpha}-g(k,\rho z+\sigma\varepsilon_{+1})}\frac{\exp(-0.5\varepsilon_{+1}^{2})}{\sqrt{2\pi}}d\varepsilon_{+1}$$

$$=\int_{-\infty}^{\infty}\frac{\beta\exp(\rho z+\sigma\sqrt{2}\tilde{\varepsilon}_{+1})\alpha k^{\alpha-1}}{\exp(\rho z+\sigma\sqrt{2}\tilde{\varepsilon}_{+1})k^{\alpha}-g(k,\rho z+\sigma\sqrt{2}\tilde{\varepsilon}_{+1})}\frac{\exp(-\tilde{\varepsilon}_{+1}^{2})}{\sqrt{\pi}}d\tilde{\varepsilon}_{+1}$$

where $\varepsilon_{+1}=\tilde{\varepsilon}_{+1}\sqrt{2}$ and the Jacobian, $\sqrt{2}$, is used when implementing the change in variables

Hermite Gaussian Quadrature

$$x \sim N(\mu, \sigma^2)$$

$$\mathsf{E}\left[H(x)\right] \approx \sum_{j=1}^{J} \left(\frac{H\left(\mu + \sqrt{2}\sigma\zeta_j\right)\omega_j}{\sqrt{\pi}} \right)$$

Hermite Gaussian Quadrature

$$\int_{-\infty}^{\infty} \frac{\beta \exp(\rho z + \sigma\sqrt{2}\tilde{\varepsilon}_{+1})\alpha k^{\alpha-1}}{\exp(\rho z + \sigma\sqrt{2}\tilde{\varepsilon}_{+1})k^{\alpha} - g(k, \rho z + \sigma\sqrt{2}\tilde{\varepsilon}_{+1})} \frac{\exp(-\tilde{\varepsilon}_{+1}^2)}{\sqrt{\pi}} d\tilde{\varepsilon}_{+1}$$

$$\approx$$

$$\sum_{j=1}^{J} \frac{\beta \exp(\rho z + \sigma\sqrt{2}\zeta_j)\alpha k^{\alpha-1}}{\exp(\rho z + \sigma\sqrt{2}\zeta_j)k^{\alpha} - g(k, \rho z + \sigma\sqrt{2}\zeta_j)} \frac{1}{\sqrt{\pi}} \omega_j$$

Euler-equation errors - Pros & Cons

1. Pro: if checked at *fine* grid then close to definition solution
2. Con: only tests for *one-period* ahead forecast errors; ignores possibility of accumulation of small errors
   - Dynamic Euler-equation error could pick those up
   - DHM statistic could pick those up

Dynamic Euler-equation errors

- Generate time series for z_t and choose k_0
- Generate two time paths for endogenous variables c_t and k_t
  1. generate time series for c_t & k_t with numerical approximation
  2. generate *alternative* series doing the following in each period
     - use numerical approx. to calculate cond. expect. accurately
     - use this conditional expectation to calculate implied consumption value
     - get capital from this implied consumption value & budget constraint
     - (numerical approximation only used to calculate cond. expect.)

Details of step 2

1. Generate time series for z_t and set $k_{imp,0} = k_0$
2. Given values of z_t and k_t: calculate conditional expectation $(= g_t)$ exactly as with regular Euler-eq. errors.
   (Thus use your numerical solution to evaluate choices inside the integral)
3. Calculate $c_{imp,t} = g^{-1/\gamma}$
4. Calculate $k_{imp,t} = z_t k_{imp,t-1}^{\alpha} + (1-\delta)k_{imp,t-1} - c_{imp,t}$

Welfare-based accuracy tests

- Be careful
- Welfare loss of using $k_t = k_{ss}\ \forall t$, instead of the optimal policy function is relatively small
  $\Longrightarrow$ different approximations can be similar in terms of welfare

DHM Accuracy test

$$\mathsf{E}\left[f(x_{t-1}, x_t, y_t, y_{t+1})|I_t\right] = 0$$

$$\Longrightarrow$$

$$\mathsf{E}\left[f(x_{t-1}, x_t, y_t, y_{t+1})h(s_t)|I_t\right] = 0$$

$$\Longrightarrow$$

$$\mathsf{E}\left[f(x_{t-1}, x_t, y_t, y_{t+1})h(s_t)'\right] = 0$$

for any $s_t \in I_t$ and any measurable function $h(\cdot)$

Use simulated data to test

$$\frac{\sum_{t=1}^{T} f(x_{t-1}, x_t, y_t, y_{t+1})h(s_t)'}{T} \approx 0$$

Simple DHM Accuracy test

1. Calculate $\bar{u}$, the average of

$$u_t = c_t^{-\gamma} - \beta c_{t+1}^{-\gamma}(\alpha \exp(z_{t+1})k_t^{\alpha-1} + 1 - \delta)$$

2. Calculate how much this error would change steady state consumption

$$\begin{aligned} c^{-\gamma} &= \bar{u} + c_{ss}^{-\gamma} \\ c &= \left(\bar{u} + c_{ss}^{-\gamma}\right)^{-1/\gamma} \end{aligned}$$

3. Express error as fraction of steady state value

$$\frac{c - c_{ss}}{c_{ss}}$$

Formal DHM Accuracy test

1. Simulate sample of T obs. (Say $T = 3,500$ & discard 500)
2. Calculate

$$J_T = TM_T' W_T^{-1} M_T$$

$$M_T = \frac{\sum_{t=1}^{T} h(s_t) f(x_{t-1}, x_t, y_t, y_{t+1})}{T}$$

$$W_T = \frac{\sum_{t=1}^{T} f(x_{t-1}, x_t, y_t, y_{t+1}) h(s_t)' h(s_t) f(x_{t-1}, x_t, y_t, y_{t+1})}{T}$$

Formal DHM Accuracy test

- J_T has a χ^2 distribution with n_h degrees of freedom
- If $h(s_t)$ is a scalar, then

$$J_T = \left(\frac{M_T}{\sqrt{W_T / T}} \right)^2$$

Implementation of DHM statistic

1. Do the DHM statistic N times
2. Check the fraction of times the statistic is in the lower and upper 5% range; inaccurate solutions are typically blown away (because of having too many realizations in the upper critical region)
3. Personally, I prefer to do the test multiple times for scalar $h(s_t)$ because this provides more information. In fact, using $h(s_t) = 1$ can already be quite informative

Limits of DHM statistic

1. Even accurate solutions are rejected more often than 5% for high enough T; thus the higher the value of T for which you get good results the better
2. Results are random so inaccurate solutions could get through by sheer chance
3. The opposite of #2 turns out to be a bigger problem in practice: DHM is often difficult to pass in the sense that solutions that in many aspects are close to the true or an extremely accurate solution can fail the DHM statistic miserably

Example - Matching model

Household side

$$\max_{\{C_t,K_t\}_{t=1}^{\infty}} \mathsf{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{C_t^{1-\gamma}-1}{1-\gamma}$$

$$\text{s.t.} \quad C_t + K_t = W_t N_{t-1} + R_t K_{t-1} + (1-\delta)K_{t-1} + P_t \tag{3}$$

$$N_t = (1-\rho^x)N_{t-1} + M_t \tag{4}$$

Household takes the number of "matches", M_t, the wage rate, W_t, the rental rate R_t, and profits, P_t, as given.
FOC

$$C_t^{-\gamma} = \mathsf{E}_t \left[\beta C_{t+1}^{-\gamma} (R_{t+1} + 1 - \delta) \right]$$

Matching model example

Problem for firm matched with worker

$$\max_{k_t} z_t k_t^{\alpha} - W_t - R_t k_t$$

FOC:

$$R_t = \alpha z_t k_t^{\alpha - 1}$$

Firm-level profits are (at optimal k) equal to

$$p_t = (1 - \alpha) z_t k_t^{\alpha} - W_t$$

Wages are given by the following rule

$$W_t = (1 - \omega_0) \times [\omega_1 * p_t + (1 - \omega_1)\bar{p}]$$

where $\bar{p}$ are steady state level profits. Wages are completely sticky if ω_1 is equal to 0.

Matching model example

Free entry

$$\text{posting cost} = \text{prob of success} \times \text{value if success}$$

$$\psi = \frac{M_t}{V_t} g_t$$

$$g_t = \mathsf{E}_t \left[\beta \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} p_{t+1} + (1 - \rho^x) g_{t+1} \right]$$

Matching model example

Matching technology

$$M_t = \frac{U_t V_t}{\left(U_t^{\xi} + V_t^{\xi}\right)^{1/\xi}}$$

with

$$U_t = 1 - N_{t-1}$$

Matching model example

Equilibrium

Equilibrium in the rental market

$$K_{t-1} = N_{t-1} k_t$$

profits transferred to households

$$P_t = N_{t-1} p_t - \psi V_t$$

Equations: Household

$$C_t^{-\gamma} = \mathsf{E}_t\left[\beta C_{t+1}^{-\gamma}(R_{t+1}+1-\delta)\right]$$

```
exp(-nu*c)=dfactor*exp(-nu*c(+1))*(exp(r(+1))+1-delta)
```

$$C_t + K_t + \psi V_t = z_t K_{t-1}^{\alpha} N_{t-1}^{1-\alpha} + (1-\delta)K_{t-1} \text{ or}$$

$$C_t + I_t + \psi V_t = Y_t,\ Y_t = z_t K_{t-1}^{\alpha} N_{t-1}^{1-\alpha},\ I_t = K_t - (1-\delta)K_{t-1}$$

```
exp(c)+exp(i)+pcost*exp(v)=exp(y)
```

```
exp(k)=(1-delta)*exp(k(-1))+exp(i)
```

```
y=varz+alpha*k(-1)+(1-alpha)*n(-1)
```

Equations: Matching

$$N_t = (1-\rho^x)N_{t-1} + \frac{U_t V_t}{\left(U_t^\xi + V_t^\xi\right)^{1/\xi}}$$

```
exp(n)
=(1-rox)*exp(n(-1))+exp(u+v)
/((exp(u*etam)+exp(v*etam))^(1/etam))
```

$$U_t = 1 - N_{t-1}$$

exp(u)=1-exp(n(-1))

Equations: rental rate & productivity

$$R_t = \alpha z_t k_t^{\alpha-1}$$

```
r=log(alpha)+varz+(alpha-1)*(k(-1)-n(-1))
```

$$\ln(z_t) = \rho \ln(z_{t-1}) + \varepsilon_t$$

```
varz=rho*varz(-1)+e
```

Equations: free entry

$$\psi = \frac{M_t}{V_t} g_t$$

```
pcost=
exp(eta)*exp(u)/((exp(u*etam)+exp(v*etam))^(1/etam))
```

$$g_t = \mathsf{E}_t \left[\beta \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} p_{t+1} + (1 - \rho^x) g_{t+1} \right]$$

```
exp(eta)=
dfactor*(exp(c(+1))/exp(c))^(-nu)
*(exp(prof(+1))+(1-rox)*exp(eta(+1)))
```

$$p_t = (1-\alpha) z_t k_t^{\alpha} - W_t$$

```
prof
=
log(
(1-omega1*omega0)*(1-alpha)*exp(varz+alpha*(k(-1)-n(-1))
-(1-omega1)*omega0*profitss
)
```

System

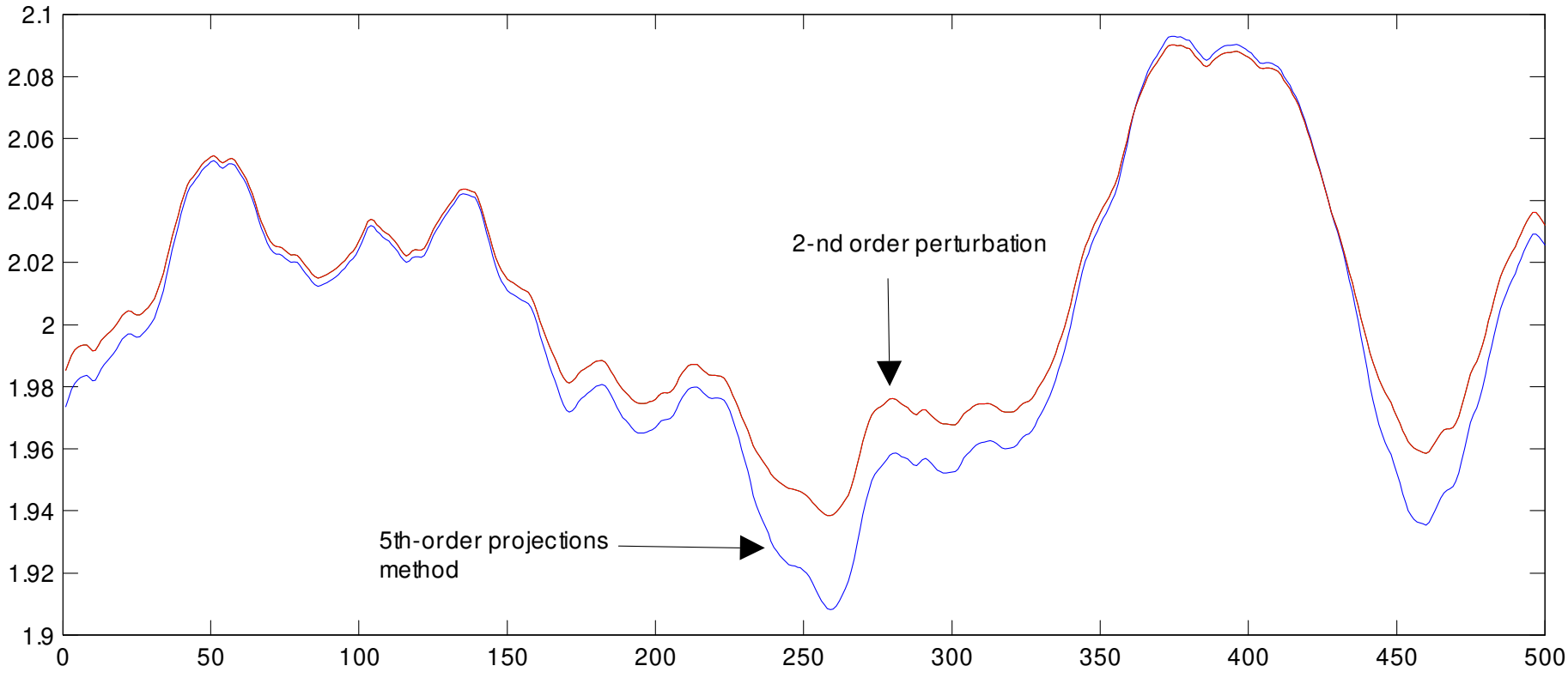
11 equations in 11 unknowns:

- $N_t, g_t, V_t, C_t, K_t, R_t, U_t, p_t, \ln(z_t), Y_t, I_t$
- n, eta, v, c, k, r, u, prof, y, varz, i

Accuracy errors

	2-nd order perturbation	5-th order projections
Capital Euler equation		
average	0.034%	0.026%
max	0.34%	0.33%
Employment Euler equation		
average	0.89%	0.004%
max	2.31%	0.086%

Log capital level



Log employment level