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Financial Frictions in DSGE Models

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1 INTRODUCTION

This course studies the incorporation of financial frictions DSGE models and their use for the design of monetary policy and financial regulation. We extend the benchmark NK model by adding a financial friction either between a bank and the firm or entrepreneur or between the household and the bank. The seminal paper in the macroeconomics literature is Bernanke *et al.* (1999), henceforth BGG. This has been developed by Gertler and Kiyotaki (2010) (henceforth, GK) in a number of respects. Both these contributions embed the friction into a RBC model, but as in Gertler and Karadi (2011) we incorporate the friction into our NK model.In the following sections we work through the details of these two models. We start with the NK model.

Our two banking models all have five types of private agents: households, wholesale and retail firms, capital producers and financial intermediaries. This disaggregation facilitates the introduction of risk transfer for instance from risk averse households to possibly risk neutral firms or 'entrepreneurs'. In addition the models have a policymaker (government) that conducts monetary and fiscal policy. Our focus is on monetary policy of both a conventional and unconventional form. Fiscal policy takes the form of exogenous government spending financed by lump-sum, non-distortionary taxes. The government is able to borrow at the riskless rate and the form of its budget constraint is irrelevant. In all models, including NK, the separation of retail firms enables us to introduce sticky prices in a straightforward manner.

A later section illustrates the structure of the two NK models with financial sectors. The differences between the models are in the location of the financial frictions, their nature, the role of net worth and the distribution of risk-taking. For the BGG model the friction is between the bank and the firm-entrepreneur. For the GK model it enters in the loan-deposit relationship with the households. GK and BGG impose an incentive compatibility constraint which, in the latter case, allows for the possibility of default. In GK households and bankers are consolidated; in BGG wholesale firms are owned by risk-neutral entrepreneurs. The role of net worth plays a central role in BGG and GK; in the former it is accumulated by the firm-entrepreneurs, in the latter by the banks.

2 THE CORE NK MODEL WITHOUT A BANKING SECTOR

We now develop an NK model with the stationarized RBC model at its core. Now we add sticky prices and nominal wages. The household sector and its supply of homogeneous is as in the RBC core. We therefore focus on the supply side and the modelling of price and wage stickiness.

2.1 HOUSEHOLDS

We choose preferences compatible with balanced growth (see King *et al.* (1988)). With external habit in consumption, household j has a single-period utility

$$U_t^j = \frac{(C_t^j - \chi C_{t-1}/(1+g_t))^{1-\sigma_c} \exp\left(\frac{(\sigma_c-1)(H_t^j)^{1+\sigma_l}}{1+\sigma_l}\right) - 1}{1-\sigma_c}; \quad \chi \in [0,1) \quad \sigma_l > 0$$

$$\rightarrow \log(C_t^j - \chi C_{t-1}/(1+g_t)) - \frac{(H_t^j)^{1+\sigma_l}}{1+\sigma_l} \text{ as } \sigma_c \rightarrow 1$$

where C_{t-1} is aggregate per capita consumption whereas with internal habit we have

$$U_t^j = \frac{(C_t^j - \chi C_{t-1}^j/(1+g_t))^{1-\sigma_c} \exp\left(\frac{(\sigma_c-1)(H_t^j)^{1+\sigma_l}}{1+\sigma_l}\right) - 1}{1-\sigma_c}; \quad \chi \in [0,1) \quad \sigma_l > 0$$

$$\rightarrow \log(C_t^j - \chi C_{t-1}^j/(1+g_t)) - \frac{(H_t^j)^{1+\sigma_l}}{1+\sigma_l} \text{ as } \sigma_c \rightarrow 1$$

Defining an instantaneous marginal utility by

$$U_{C,t} = (C_t - \chi C_{t-1}/(1+g_t))^{-\sigma_c} \exp\left(\frac{(\sigma_c-1)H_t^{1+\sigma_l}}{1+\sigma_l}\right)$$

Then in a symmetric equilibrium he household first-order conditions for external habit and internal habit respectively are

$$1 = \mathbb{E}_t\left[R_{t+1}\Lambda_{t,t+1}\right]$$

$$
\begin{aligned}
\Lambda_{t,t+1} &= \beta_{g,t+1}\frac{\lambda_{t+1}}{\lambda_t} \\
\beta_{g,t} &= \beta\,(1+g_t)^{-\sigma_c} \\
U_{H,t} &= -H_t^{\sigma_l}(C_t-\chi C_{t-1}/(1+g_t))^{1-\sigma_c}\exp\left(\frac{(\sigma_c-1)H_t^{1+\sigma_l}}{1+\sigma_l}\right) \\
\frac{U_{H,t}}{\lambda_t} &= -W_t
\end{aligned}
$$

where for external habit and internal habit respectively we have

$$
\begin{aligned}
&\lambda_t = U_{C,t} \\
&\lambda_t = U_{C,t} - \beta\chi\mathbb{E}_t[U_{C,t+1}]
\end{aligned}
$$

Parameter σ_l is referred to by Smets and Wouters (2007a) as the labour supply elasticity. For the log-utility case σ_l is the Frisch parameter. SW assume a prior mean of 2 for σ_l.

2.2 STICKY PRICES

First we introduce a retail sector producing differentiated goods under monopolistic competition. This sector converts homogeneous output from a competitive wholesale sector. The aggregate prices in the two sectors are given by P_t and P_t^W respectively and $P_t > P_t^W$ from the *mark-up* possible under monopolistic competition. The *real marginal cost* of producing each differentiated good $MC_t \equiv \frac{P_t^W}{P_t}$. In the RBC model $P_t = P_t^W$ so $MC_t = 1$ and the *marginal cost is constant*. In the NK model retailers are locked into price-contracts and cannot their prices every period. Their marginal costs therefore vary. In periods of high demand they simply increase output until they are able to change prices.

The retail sectors then uses a homogeneous wholesale good to produce a basket of differentiated goods for consumption

$$
C_t = \left(\int_0^1 C_t(m)^{(\zeta-1)/\zeta} dm\right)^{\zeta/(\zeta-1)} \tag{2.1}
$$

where $\zeta > 1$ is the elasticity of substitution. For each m, the consumer chooses $C_t(m)$ at a price $P_t(m)$ to maximize (2.1) given total expenditure $\int_0^1 P_t(m)C_t(m)dm$.

This results in a set of consumption demand equations for each differentiated good m with price $P_t(m)$ of the form

$$C_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} C_t \tag{2.2}$$

where $P_t = \left[\int_0^1 P_t(m)^{1-\zeta} dm\right]^{\frac{1}{1-\zeta}}$. P_t is the aggregate price index. Note that C_t and P_t are Dixit-Stiglitz aggregators – see Dixit and Stiglitz (1977). Demand for investment and government services takes the same form (see Appendix 2) so in aggregate

$$Y_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} Y_t \tag{2.3}$$

Following Calvo (1983), we now assume that there is a probability of $1 - \xi_p$ at each period that the price of each retail good m is set optimally to $P_t^0(m)$. If the price is not re-optimized, then it is held fixed.[1] For each retail producer m, given its real marginal cost $MC_t = \frac{P_t^W}{P_t}$, the objective is at time t to choose $\{P_t^0(m)\}$ to maximize discounted real profits

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi_p^k \frac{\Lambda_{t,t+k}}{P_{t+k}} Y_{t+k}(m) \left[P_t^0(m) - P_{t+k} MC_{t+k}\right] \tag{2.4}$$

subject to

$$Y_{t+k}(m) = \left(\frac{P_t^O(m)}{P_{t+k}}\right)^{-\zeta} Y_{t+k} \tag{2.5}$$

where $\Lambda_{t,t+k} \equiv \beta^k \frac{U_{C,t+k}}{U_{C,t}}$ is the (non-stationarized) stochastic discount factor[2] over the interval $[t, t+k]$. The solution to this optimization problem is

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi_p^k \frac{\Lambda_{t,t+k}}{P_{t+k}} Y_{t+k}(m) \left[P_t^0(m) - \frac{1}{(1-1/\zeta)} P_{t+k} MC_{t+k}\right] = 0 \tag{2.6}$$

Using (2.5) and rearranging this leads to

$$P_t^O = \frac{1}{(1-1/\zeta)} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \xi_p^k \frac{\Lambda_{t,t+k}}{P_{t+k}} \left(P_{t+k}\right)^{\zeta} Y_{t+k} MC_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} \xi_p^k \frac{\Lambda_{t,t+k}}{P_{t+k}} \left(P_{t+k}\right)^{\zeta} Y_{t+k}} \tag{2.7}$$

1 Thus we can interpret $\frac{1}{1-\xi_p}$ as the average duration for which prices are left unchanged.
2 We stationarize the model later.

where the m index is dropped as all firms face the same marginal cost so the right-hand side of the equation is independent of firm size or price history.

By the law of large numbers the evolution of the price index is given by

$$P_t^{1-\zeta} = \xi_p P_{t-1}^{1-\zeta} + (1-\xi_p)(P_t^0)^{1-\zeta} \tag{2.8}$$

Now define k periods ahead inflation as

$$\Pi_{t,t+k} \equiv \frac{P_{t+k}}{P_t}$$

To ease the notation in what follows we denote $\Pi_t = \Pi_{t-1,t}$ and $\Pi_{t+1} = \Pi_{t,t+1}$.

We can now write the fraction (2.7)

$$\frac{P_t^O}{P_t} = \frac{1}{(1-1/\zeta)} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \xi_p^k \Lambda_{t,t+k} \left(\Pi_{t,t+k}\right)^{\zeta} Y_{t+k} MC_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} \xi_p^k \Lambda_{t,t+k} \left(\Pi_{t,t+k}\right)^{\zeta-1} Y_{t+k}} \tag{2.9}$$

and (2.8) as

$$1 = \xi_p \left(\Pi_t\right)^{\zeta-1} + (1-\xi_p) \left(\frac{P_t^O}{P_t}\right)^{1-\zeta} \tag{2.10}$$

2.2.1 Price Dynamics

In order to set up the model in non-linear form as a set of difference equations, required for software packages such a Dynare, we need to represent the price dynamics as *difference equations*. Both numerator and denominator the first-order condition for pricing, (2.9), are of the form considered in Appendix 2.

First we assume a zero-growth steady state so that we do not yet need to stationarize any variables. Then using the Lemma in that section , price dynamics are given by

$$\frac{P_t^0}{P_t} = \frac{J_t^p}{JJ_t^p} \tag{2.11}$$

$$JJ_t^p - \xi_p \mathbb{E}_t[\Lambda_{t,t+1} \Pi_{t+1}^{\zeta-1} JJ_{t+1}^p] = Y_t \tag{2.12}$$

$$J_t^p - \xi_p \mathbb{E}_t[\Lambda_{t,t+1} \Pi_{t+1}^{\zeta} J_{t+1}^p] = \left(\frac{1}{1-\frac{1}{\zeta}}\right) Y_t MC_t MCS_t \tag{2.13}$$

$$1 = \xi_p \Pi_t^{\zeta-1} + (1-\xi_p) \left(\frac{J_t^p}{JJ_t^p}\right)^{1-\zeta} \tag{2.14}$$

$$MC_t = \frac{P_t^W}{P_t} = \frac{W_t}{F_{H,t}} \tag{2.15}$$

where (2.15) allows for $P_t \neq P_t^W$. We have also introduced a mark-up shock MCS_t to MC_t. Notice that the real marginal cost, MC_t, is no longer fixed as it was in the RBC model.

2.2.2 Indexing

Prices are now are indexed to last period's aggregate inflation, with a price indexation parameter γ_p. Then the price trajectory with no re-optimization is given by $P_t^O(j)$, $P_t^O(j)\left(\frac{P_t}{P_{t-1}}\right)^{\gamma_p}$, $P_t^O(j)\left(\frac{P_{t+1}}{P_{t-1}}\right)^{\gamma_p}$, $\cdots$. where $Y_{t+k}(m)$ is given by (2.5) with indexing so that

$$Y_{t+k}(m) = \left(\frac{P_t^O(m)}{P_{t+k}}\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_p}\right)^{-\zeta} Y_{t+k} \tag{2.16}$$

With indexing by an amount $\gamma_p \in [0,1]$ and an exogenous mark-up shock MS_t as before, the optimal price-setting first-order condition for a firm j setting a new optimized price $P_t^0(j)$ is now given by

$$P_t^0 = \frac{\frac{\zeta}{\zeta-1}\mathbb{E}_t\left[\sum_{k=0}^{\infty}\xi_p^k\frac{\Lambda_{t,t+k}}{P_{t+k}}MC_{t+k}MS_{p,t+k}Y_{t+k}\right]}{\mathbb{E}_t\left[\sum_{k=0}^{\infty}\xi_p^k\frac{\Lambda_{t,t+k}}{P_{t+k}}Y_{t+k}(j)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma}\right]}. \tag{2.17}$$

Price dynamics are now given by

$$\begin{aligned}
\frac{P_t^0}{P_t} &= \frac{J_t^p}{JJ_t^p} \\
JJ_t^p - \xi_p\mathbb{E}_t[\Lambda_{t,t+1}\tilde{\Pi}_{t+1}^{\zeta-1}JJ_{t+1}^p] &= Y_t \\
J_t^p - \xi_p\mathbb{E}_t[\Lambda_{t,t+1}\tilde{\Pi}_{t+1}^{\zeta}J_{t+1}^p] &= \frac{\zeta}{\zeta-1}MC_tMS_{p,t}Y_t \\
\tilde{\Pi}_t &\equiv \frac{\Pi_t}{\Pi_{t-1}^{\gamma_p}} \\
1 &= \xi_p\tilde{\Pi}_t^{\zeta-1} + (1-\xi_p)\left(\frac{J_t^p}{JJ_t^p}\right)^{1-\zeta}
\end{aligned}$$

An alternative model of indexing assumes that prices are indexed to a weighted average of last period and trend (steady state) inflation. If we denote the two weights by γ_p and $\bar{\gamma}_p$ then the previous dynamics replaces $\tilde{\Pi}_t$ above with

$$\tilde{\Pi}_t \equiv \frac{\Pi_t}{\Pi_{t-1}^{\gamma_p}\Pi^{1-\bar{\gamma}_p}} \tag{2.18}$$

In Smets and Wouters (2007a) it is assumed that $\bar{\gamma}_p = \gamma_p$ so that $\tilde{\Pi} = 1$ in the steady state which eliminates the effect of state-state inflation in the equilibrium. In the coding of our model we allow for options $\bar{\gamma}_p = \gamma_p$ and $\bar{\gamma}_p = 0$.

2.2.3 Price Dynamics in a Non-Zero-Growth Steady State

Stationarizing J_t^p and JJ_t^p as in the RBC model, price dynamics with indexing become

$$\begin{aligned}
\frac{P_t^0}{P_t} &= \frac{J_t^p}{JJ_t^p} \\
JJ_t^p - \xi_p \mathbb{E}_t[(1+g_{t+1})\Lambda_{t,t+1}\tilde{\Pi}_{t+1}^{\zeta-1} JJ_{t+1}^p] &= Y_t \\
J_t^p - \xi_p \mathbb{E}_t[(1+g_{t+1})\Lambda_{t,t+1}\tilde{\Pi}_{t+1}^{\zeta} J_{t+1}^p] &= \frac{\zeta}{\zeta-1} MC_t MS_{p,t} Y_t \\
\tilde{\Pi}_t &\equiv \frac{\Pi_t}{\Pi_{t-1}^{\gamma_p}} \\
1 &= \xi_p \tilde{\Pi}_t^{\zeta-1} + (1-\xi_p)\left(\frac{J_t^p}{JJ_t^p}\right)^{1-\zeta}
\end{aligned}$$

2.3 STICKY WAGES

To introduce wage stickiness we now assume that each household supplies homogeneous labour at a nominal wage rate $W_{h,t}$ to a monopolistic trade-union who differentiates the labour and sells type $H_t(j)$ at a nominal wage $W_{n,t}(j) > W_{h,t}$ to a labour packer in a sequence of Calvo staggered nominal wage contracts. The real wage is then defined as $W_t \equiv \frac{W_{n,t}}{P_t}$. We now have to distinguish between *price inflation* which now uses the notation $\Pi_t^p \equiv \frac{P_t}{P_{t-1}}$ and *wage inflation*, $\Pi_t^w \equiv \frac{W_{n,t}}{W_{n,t-1}} = \frac{W_t \Pi_t^p}{W_{t-1}}$.

As with price contracts we employ Dixit-Stiglitz quantity and price aggregators. Calvo probabilities are now ξ_p and ξ_w for price and wage contracts respectively. The competitive labour packer forms a composite labour service according to $H_t = \left(\int_0^1 H_t(j)^{(\mu-1)/\mu} dj\right)^{\mu/(\mu-1)}$ and sells onto the intermediate firm. where $\mu > 1$ is the elasticity of substitution. For each j, the labour packer chooses $H_t(j)$ at a wage $W_{n,t}(j)$ to maximize H_t given total expenditure $\int_0^1 W_{n,t}(j)H_t(j)dj$. This results in a set of labour demand equations for each differentiated labour type j with wage $W_{n,t}(j)$ of the form

$$H_t(j) = \left(\frac{W_{n,t}(j)}{W_{n,t}}\right)^{-\mu} H_t \tag{2.19}$$

where $W_{n,t} = \left[\int_0^1 W_{n,t}(j)^{1-\mu} dj\right]^{\frac{1}{1-\mu}}$ is the aggregate nominal wage index. H_t and $W_{n,t}$ are Dixit-Stigliz aggregators for the labour market.

Wage setting by the trade-union again follows the standard Calvo framework supplemented with indexation. At each period there is a probability $1-\xi_w$ that the wage is set optimally. The optimal wage derives from maximizing discounted profits. For those trade-unions unable to reset, wages are indexed to last period's aggregate inflation, with wage indexation parameter γ_w. Then as for price contracts the wage rate trajectory with no re-optimization is given by $W^O_{n,t}(j)$, $W^O_{n,t}(j)\left(\frac{P_t}{P_{t-1}}\right)^{\gamma_w}$, $W^O_{n,t}(j)\left(\frac{P_{t+1}}{P_{t-1}}\right)^{\gamma_w}$, $\cdots$. The trade union then buys homogeneous labour at a nominal price $W_{h,t}$ and converts it into a differentiated labour service of type j. The trade union time t then chooses $W^O_{n,t}(j)$ to maximize real profits

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi_w^k \frac{\Lambda_{t,t+k}}{P_{t+k}} H_{t+k}(j) \left[W^O_{n,t}(j)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_w} - W_{h,t+k}\right] \tag{2.20}$$

where using (2.19) with indexing $H_{t+k}(j)$ is given by

$$H_{t+k}(j) = \left(\frac{W^O_{n,t}(j)}{W_{n,t+k}}\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_w}\right)^{-\mu} H_{t+k} \tag{2.21}$$

and μ is the elasticity of substitution across labour varieties.

By analogy with (2.6) this leads to the following first-order condition

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi_w^k \frac{\Lambda_{t,t+k}}{P_{t+k}} H_{t+k}(j) \left[W^0_t(j)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_w} - \frac{\mu}{\mu-1} W_{h,t+k}\right] = 0$$

(2.22)

and hence by analogy with (2.17) this leads to the optimal real wage

$$\frac{W^O_{n,t}}{P_t} = \frac{\mu}{\mu-1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \xi_w^k \Lambda_{t,t+k} \left(\Pi^w_{t,t+k}\right)^{\zeta} H_{t+k} \frac{W_{h,t+k}}{P_{t+k}}}{\mathbb{E}_t \sum_{k=0}^{\infty} \xi_w^k \Lambda_{t,t+k} \left(\Pi^w_{t,t+k}\right)^{\zeta} \left(\Pi^p_{t,t+k}\right)^{-1} H_{t+k}} = \frac{J^w_t}{JJ^w_t} \tag{2.23}$$

Then by the law of large numbers the evolution of the wage index is given by

$$W_{n,t}^{1-\mu} = \xi_w \left(W_{n,t-1} \left(\frac{P_t}{P_{t-1}} \right)^{\gamma_w} \right)^{1-\mu} + (1-\xi_w)(W^0_{n,t}(j))^{1-\mu} \tag{2.24}$$

Finally to facilitate the introduction of banks that allow entry and exit into this sector by households we introduce *capital producers* following Section 1.1 in the Appendix.

2.3.1 Dynare Price and Wage Dynamics

We now apply the analysis of 2.2.1-2.2.3 to wage dynamics and bring the two forms together. The model is now stationarized.

$$\begin{aligned}
\Pi^p_t &\equiv \frac{P_t}{P_{t-1}} \\
\tilde{\Pi}^p_t(\gamma) &\equiv \frac{\Pi^p_t}{\Pi^{\gamma}_{p,t-1}} \\
JJ^p_t - \xi_p \mathbb{E}_t[(1+g_{t+1})\Lambda_{t,t+1}\tilde{\Pi}^p_{t+1}(\gamma_p)^{\zeta-1} JJ^p_{t+1}] &= Y_t \\
J^p_t - \xi_p \mathbb{E}_t[(1+g_{t+1})\Lambda_{t,t+1}\tilde{\Pi}^p_{t+1}(\gamma_p)^{\zeta} J^p_{t+1}] &= \frac{\zeta}{\zeta-1} Y_t MC_t MS_{p,t} \\
1 &= \xi_p \tilde{\Pi}^p_t(\gamma_p)^{\zeta-1} + (1-\xi_p)\left(\frac{J^p_t}{JJ^p_t}\right)^{1-\zeta} \\
\frac{P^O_t}{P_t} &= \frac{J^p_t}{JJ^p_t}
\end{aligned}$$

$$\Pi^w_t \equiv \frac{W_{n,t}}{W_{n,t-1}} = (1+g_t)\frac{\Pi_t W_t}{W_{t-1}} \tag{2.25}$$

$$\tilde{\Pi}^w_t \equiv \frac{\Pi^w_t}{(\Pi^p_{t-1})^{\gamma_w}} \tag{2.26}$$

$$\begin{aligned} MRS_t &= -\frac{U_{H,t}}{U_{C,t}} = \frac{W_{h,t}}{P_t} && (2.27)\\ JJ_t^w - \xi_w \mathbb{E}_t\left[\Lambda_{t,t+1}\frac{(\tilde{\Pi}^w_{t,t+1})^{\mu}}{\tilde{\Pi}^p_{t,t+1}(\gamma_w)} JJ^w_{t+1}\right] &= H_{d,t} && (2.28)\\ J_t^w - \xi_w \mathbb{E}_t\left[(1+g_{t+1})\Lambda_{t,t+1}\tilde{\Pi}^{\mu}_{w,t+1} J^w_{t+1}\right] &= -\frac{\mu}{\mu-1} MRS_t MS_{w,t} H_{d,t} && (2.29) \end{aligned}$$

$$\begin{aligned} (W_{n,t})^{1-\mu} &= \xi_w\left((W_{n,t-1})\frac{1}{\tilde{\Pi}^p_t(\gamma_w)}\right)^{1-\mu} + (1-\xi_w)\left(W^O_{n,t}\right)^{1-\mu} \Rightarrow\\ 1 &= \xi_w\left(\frac{\Pi^w_t\tilde{\Pi}_{p,t}(\gamma_w)}{\Pi^p_t}\right)^{\mu-1} + (1-\xi_w)\left(\frac{W^O_{n,t}(j)}{W_{n,t}}\right)^{1-\mu} && (2.30)\\ W^O_t &\equiv \frac{W^O_{n,t}}{W_{n,t}} = \frac{W^O_{n,t}/P_t}{W_{n,t}/P_t} = \frac{J^w_t}{W_t JJ^w_t} && (2.31)\\ \Pi^w_t &= (1+g_t)\frac{\Pi_t W_t}{W_{t-1}} && (2.32) \end{aligned}$$

2.4 CAPACITY UTILIZATION AND FIXED COSTS OF PRODUCTION

We now add two remaining features to the model. As in Christiano *et al.* (2005) and Smets and Wouters (2007a) we assume that using the stock of capital with intensity u_t produces a cost of $a(u_t)K_t$ units of the composite final good. The functional form is chosen consistent with the literature:

$$a(u_t) = \gamma_1(u_t-1) + \frac{\gamma_2}{2}(u_t-1)^2 \qquad (2.33)$$

and satisfies $a(1) = 0$ and $a'(1), a''(1) > 0$. Then we must add a term $(r^K_t - a(u_t)K_t$ to the household budget constraint on the income side where r^K_t is the rental rate leading to the following first-order condition determines capacity utilization:

$$r^K_t = a'(u_t) \qquad (2.34)$$

Capital now enters the production function as $u_t K_{t-1}$.

The final change is to add fixed costs F, necessary to transform homogeneous wholesale goods into differentiated retail goods. To pin down F we make the assumption that entry occurs until retail profits are eliminated in the steady state , i.e., $P^W Y^W = PY$. It follows that

$$\frac{P^W}{P} = MC = \frac{Y}{Y^W} = \frac{\left(1 - \frac{F}{Y^W}\right)}{\Delta_p} \tag{2.35}$$

It follow that

$$\frac{F}{Y^W} = 1 - \Delta_p MC \tag{2.36}$$

For the zero inflation, $MC = 1 - \frac{1}{\zeta}$ and $\Delta_p = \Delta_w = 1$ and therefore $\frac{F}{Y^W} = \frac{1}{\zeta}$.

Smets and Wouters (2007a) has one more feature: *Kimball preferences* as in Kimball (1995) and Klenow and Willis (2016) generalize the Dixit-Stigliz aggregator.

2.5 PRICE AND WAGE DISPERSION

The output and labour market clearing conditions must take into account relative price dispersion across varieties and wage dispersion across firms. Integrating across all firms, taking into account that the capital-labour ratio is common across firms and that the wholesale sector is separated from the retail sector we obtain aggregate demand for intermediate (wholesale) goods necessary to produce final retail goods as

$$Y_t^W - F = \int_0^1 \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} dm(C_t + I_t + G_t) = \Delta_t^p Y_t \tag{2.37}$$

where labour market clearing gives total demand for labour, H_t^d, as

$$H_t = \int_0^1 H_t(j)dj = \int_0^1 \left(\frac{W_{n,t}(j)}{W_{n,t}}\right)^{-\mu} dj\, H_t^d = \Delta_t^w H_t^d \tag{2.38}$$

where the price dispersion is given by $\Delta_t^p = \int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{-\zeta} df$ and wage dispersion is given by $\Delta_t^w = \int_0^1 \left(\frac{W_{n,t}(j)}{W_{n,t}}\right)^{-\mu} dj$. From Section 2.3 in the Appendix for price dispersion and by analogy for wage dispersion we have:

$$\Delta_t^p = \xi_p + \tilde{\Pi}_t^{\zeta} \Delta_{t-1}^p + (1 - \xi_p)\left(\frac{P_t^O}{P_t}\right)^{-\zeta} \tag{2.39}$$

$$\Delta_t^w = \xi_w \tilde{\Pi}_{w,t}^{\mu} \Delta_{t-1}^w + (1 - \xi_w)\left(\frac{W_{n,t}^O}{W_{n,t}}\right)^{-\mu} \tag{2.40}$$

2.6 A BALANCED-NON-ZERO-GROWTH STEADY STATE

We can easily set up the model with a balanced-exogenous-growth steady state. Now the process for A_t is replaced with

$$\begin{aligned} A_t &= \bar{A}_t A_t^c \\ \bar{A}_t &= (1+g_t)\bar{A}_{t-1} \\ \log(1+g_t) &= \log(1+g) + \epsilon_{A,t} \\ \log A_t^c - \log A^c &= \rho_A(\log A_{t-1}^c - \log A^c) + \epsilon_{A,t} \end{aligned}$$

A_t is a labour-augmenting technical progress parameter which is now decomposed into a cyclical component, A_t^c, modelled as a temporary AR1 process, a stochastic trend, a random walk with drift, $\bar{A}_t$. Thus the balanced growth deterministic steady state path (bgp) is driven by labour-augmenting technical change growing at a net rate g. If we put $g = \epsilon_{trend,t} = 0$ and $\bar{A}_t = 1$, we arrive at our previous formulation with $A_t^c = A_t$.

$$\begin{aligned} U_t^c &= \frac{((C_t^c - \chi C_{t-1}^c/(1+g_t))^{(1-\varrho)}(1-H_t)^{\varrho})^{1-\sigma_c} - 1}{1-\sigma_c} \\ U_{C,t}^c &\equiv \frac{U_{C,t}}{\bar{A}_t^{(1-\varrho)(1-\sigma_c)-1}} = (1-\varrho)(C_t^c - \chi C_{t-1}^c/(1+g_t))^{(1-\varrho)(1-\sigma_c)-1}(1-H_t)^{\varrho(1-\sigma_c)} \\ \Lambda_{t,t+1} &= \beta\frac{U_{C,t+1}}{U_{C,t}} = \beta(1+g_{t+1})^{(1-\varrho)(1-\sigma_c)-1}\frac{U_{C,t+1}^c}{U_{C,t}^c} \equiv \beta_{g,t+1}\frac{U_{C,t+1}^c}{U_{C,t}^c} \end{aligned}$$

where the growth-adjusted discount rate is defined as

$$\beta_{g,t} \equiv \beta(1+g_t)^{(1-\varrho)(1-\sigma_c)-1},$$

the Euler equation is still

$$E_t\left[\Lambda_{t,t+1}R_{t+1}\right]$$

Now stationarize remaining variables by defining cyclical components:

$$\begin{aligned} Y_t^c &\equiv \frac{Y_t}{\bar{A}_t} = \frac{(A_t H_t^d)^{\alpha}\left(\frac{K_{t-1}}{\bar{A}_t}\right)^{1-\alpha} - F}{\Delta_t^p} = \frac{(A_t H_t^d)^{\alpha}\left(\frac{K_{t-1}^c}{(1+g_t)}\right)^{1-\alpha} - F}{\Delta_t^p} \\ K_t^c &\equiv \frac{K_t}{\bar{A}_t} \end{aligned}$$

$$
\begin{aligned}
K_t^c &= (1-\delta)\frac{K_{t-1}^c}{1+g_t} + (1 - S(X_t^c))I_t^c \\
X_t^c &= (1+g_t)\frac{I_t^c}{I_{t-1}^c} \\
S(X_t^c) &= \phi_X(X_t^c - 1 - g_t)^2 \\
S'(X_t^c) &= 2\phi_X(X_t^c - 1 - g_t) \\
C_t^c &\equiv \frac{C_t}{\bar{A}_t} \\
I_t^c &\equiv \frac{I_t}{\bar{A}_t} \\
W_t^c &\equiv \frac{W_t}{\bar{A}_t}
\end{aligned}
$$

With non-zero steady state growth, the steady state for the rest of the system is the same as the zero-growth one except for the following relationships:

$$
\begin{aligned}
R &= \frac{(1+g)^{1+(\sigma_c-1)(1-\varrho)}}{\beta} \equiv \frac{1}{\beta_g} \\
R_n &= \Pi R \\
\Lambda &= \frac{1}{R} \\
I^c &= \frac{(\delta+g)K^c}{1+g} \\
X^c &= 1+g \\
\frac{(J^p)^c}{(JJ^p)^c} &= \left(\frac{1-\xi_p(\Pi^p)^{\zeta-1}}{1-\xi_p}\right)^{\frac{1}{1-\zeta}} \\
MC &= \left(1-\frac{1}{\zeta}\right)\frac{J^p(1-\beta_g(1+g)\xi_p\Pi^\zeta)}{JJ^p(1-\beta_g(1+g)\xi(\Pi^p)^{\zeta-1})} \\
\frac{\frac{W_h}{P}}{\frac{W}{P}} &= \left(1-\frac{1}{\mu}\right)\frac{\frac{J^w}{JJ^w}}{\frac{W}{P}}\frac{(1-\beta_g\xi_w\tilde{\Pi}^p(\gamma_w)^\mu}{(1-\beta_g\xi_w\tilde{\Pi}_p(\gamma_w)^{\mu-1})} \\
&= \text{Inverse of wage mark-up}
\end{aligned}
$$

where R and R_n are the real and nominal steady state interest rates and Π is inflation.

2.7 SUMMARY OF SUPPLY SIDE AND EXPECTED SPREAD

Wholesale, Retail and capital producer firm behaviour is given by

$$\textbf{Wholesale Production} \quad : \quad Y_t^W = (A_t H_t^d)^\alpha K_{t-1}^{1-\alpha} \tag{2.41}$$

$$\textbf{Retail Aggregate Production} \quad : \quad Y_t = \frac{Y_t^W - F}{\Delta_t^p} \tag{2.42}$$

$$\textbf{Aggregate Employed Labour} \quad : \quad H_t^d = \frac{H_t}{\Delta_t^w} \tag{2.43}$$

$$\textbf{Labour Demand} \quad : \quad W_t = \frac{P_t^W}{P_t} F_{H,t} = \frac{P_t^W}{P_t} \frac{\alpha Y_t^W}{H_t^d} \tag{2.44}$$

$$\textbf{Capital Demand} \quad : \quad r_t^K = \frac{P_t^W}{P_t} F_{K,t} = \frac{P_t^W}{P_t} \frac{(1-\alpha) Y_t^W}{K_{t-1}} \tag{2.45}$$

where K_t is *end-of-period* $[t, t+1]$ capital, W_t is the wage rate of the composite differentiated labour orovided by the labour packer (trade-union) and Δ_t^p and Δ_t^w are price dispersion and wage dispersion (defined below), r_t^K is the rental net rate for capital and we have imposed labour demand equal to labour supply in a labour market equilibrium. Production is assumed to be Cobb-Douglas.

Capital accumulation with investment adjustment costs carried out by **Capital Producers** is given by

$$K_t = (1-\delta) K_{t-1} + (1 - S(X_t)) I_t IS_t \tag{2.46}$$

$$X_t \equiv \frac{I_t}{I_{t-1}} \tag{2.47}$$

$$S(X_t) = \phi_X (X_t - 1 - g)^2 \tag{2.48}$$

$$S'(X_t) = 2\phi_X (X_t - 1) \tag{2.49}$$

$$Q_t IS_t (1 - S(X_t) - X_t S'(X_t)) + \mathbb{E}_t \left[\Lambda_{t,t+1} Q_{t+1} IS_{t+1} S'(X_{t+1}) X_{t+1}^2 \right] = 1 \tag{2.50}$$

where I_t, and Q_t are investment and the real price of capital respectively. IS_t is a capital specific shock process. $S(X_t)$ are investment adjustment costs equal to zero in a balance growth steady state with output, consumption, capital, investment and the real wage growing at a rate g.

Then this completes the supply side with price and wage dynamics and dispersion as given in sections 2.3.1 and 2.5.

2.7.1 Expected Spread

The gross return on capital by

$$R_t^K = \left[\frac{r_t^K + (1-\delta)Q_t}{Q_{t-1}}\right] \tag{2.51}$$

Then in the *absence of financial frictions* including the risk-premium shock RPS_t we have *arbitrage* between discounted returns on capital and bonds given by

$$\mathbb{E}_t[\Lambda_{t,t+1}R_{t+1}^K] = \mathbb{E}_t[\Lambda_{t,t+1}R_{t+1}] = 1 \tag{2.52}$$

With the risk premium shock, $\mathbb{E}_t[\Lambda_{t,t+1}R_{t+1}RPS_t] = 1$ and the expected discounted spread between returns on capital and bonds becomes $\mathbb{E}_t[\Lambda_{t,t+1}(R_{t+1}^K - R_{t+1})] \neq 0$ with a steady state given by $RK - R = \frac{1}{\Lambda}(1 - \frac{1}{RPS}) = \frac{1}{\beta_g}(1 - \frac{1}{RPS})$. A postive steady state of the premium shock can then be chosen to be a value $RPS > 1$ to fit data on the mean spread, $\overline{spread}$ by putting

$$RPS = \frac{1}{1 - \beta_g \overline{spread}} \tag{2.53}$$

2.8 THE MONETARY RULE, OUTPUT EQUILIBRIUM AND SHOCKS

The nominal interest rate is given by one of the following Taylor-type rules

$$\begin{aligned}\log\left(\frac{R_{n,t}}{R_n}\right) &= \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1-\rho_r)\Big[\theta_\pi \log\left(\frac{\Pi_t}{\Pi}\right) \\ &+ \theta_y \log\left(\frac{Y_t}{Y}\right) + \theta_{dy}\log\left(\frac{Y_t}{Y_{t-1}}\right)\Big] + \epsilon_{MPS,t}\end{aligned} \tag{2.54}$$

$$\begin{aligned}\log\left(\frac{R_{n,t}}{R_n}\right) &= \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1-\rho_r)\Big[\theta_\pi \log\left(\frac{\Pi_t}{\Pi}\right) \\ &+ \theta_y \log\left(\frac{Y_t}{Y_t^F}\right) + \theta_{dy}\log\left(\frac{Y_t/Y_t^F}{Y_{t-1}/Y_{t-1}^F}\right)\Big] + \epsilon_{MPS,t}\end{aligned} \tag{2.55}$$

where Y_t^F is the flexi-price level of output and $\epsilon_{M,t}$ is a monetary policy shock process. (2.54) is an 'implementable' form of the Taylor rule which stabilizes output about its steady state. Then θ_π and θ_y are the long-run elasticities of the inflation and output respectively with respect to the interest rate. The "Taylor principle" requires $\theta_\pi > 1$. The conventional Taylor rule, (2.55), stabilizes output about its flexi-price level which is that found by solving the RBC core of this model or simply allowing the contract parameter ξ_p to tend to zero. Unlike the implementable form, this requires observations of the output gap $\frac{Y_t}{Y_t^F}$ to implement and monitor.[3] The output equilibrium is given by

$$Y_t = C_t + G_t + I_t \tag{2.56}$$

Finally the model is closed with seven exogenous AR1 shock processes to technology, government spending, the real marginal cost (the latter being interpreted as a mark-up shock), the marginal rate of substitution, an investment shock, a risk premium shock and a shock to monetary policy

$$\begin{aligned}
\log A_t - \log A &= \rho_A(\log A_{t-1} - \log A) + \epsilon_{A,t} \\
\log G_t - \log G &= \rho_G(\log G_{t-1} - \log G) + \epsilon_{G,t} \\
\log MS_t - \log MS &= \rho_{MS}(\log MS_{t-1} - \log MS) + \epsilon_{MS,t} \\
\log MRSS_t - \log MRSS &= \rho_{MRSS}(\log MRSS_{t-1} - \log MRSS) + \epsilon_{MRSS,t} \\
\log IS_t - \log IS &= \rho_{IS}(\log IS_{t-1} - \log IS) + \epsilon_{IS,t} \\
\log RPS_t - \log RPS &= \rho_{RPS}(\log RPS_{t-1} - \log RPS) + \epsilon_{RPS,t} \\
\log MPS_t - \log MPS &= \rho_{MPS}(\log MPS_{t-1} - \log MPS) + \epsilon_{MPS,t}
\end{aligned}$$

2.9 FULL NK MODEL LISTING:

The full model in stationarized form is given by:

2.9.1 Dynamic Model

$$\begin{aligned}
\log(1+g_t) &= \log(1+g) + \epsilon_{Atrend,t} \\
\beta_{g,t} &= \beta\,(1+g_t)^{-\sigma_c} \\
U_t &= \frac{(C_t - \chi C_{t-1}/(1+g_t))^{1-\sigma_c} \exp\left(\frac{(\sigma_c-1)(H_t)^{1+\sigma_l}}{1+\sigma_l}\right) - 1}{1-\sigma_c}
\end{aligned}$$

3 Technically this should pose no problems in a perfect information rational expectations equilibrium, but the rationale for 'simple rules' is to have policies that are easy to observe without relying on the perfect information solution.

$$
\begin{aligned}
CE_t &= \frac{(1.01(C_t - \chi C_{t-1}/(1+g_t)))^{1-\sigma_c} \exp\left(\frac{(\sigma_c - 1)(H_t)^{1+\sigma_l}}{1+\sigma_l}\right) - 1}{1-\sigma_c} - U_t \\
&+ \mathbb{E}_t[(1+g_{t+1})\ \beta_{g,t+1}\ CE_{t+1}] \\
\Omega_t &= U_t + \beta\, \mathbb{E}_t[(1+g)^{1-\sigma_c}\ exp\,((1-\sigma_c)\ \epsilon_{Atrend,t+1})\ \Omega_{t+1}] \\
U_{C_t} &= (C_t - \chi C_{t-1}/(1+g_t))^{-\sigma_c} \exp\left(\frac{(\sigma_c - 1) H_t^{1+\sigma_l}}{1+\sigma_l}\right) \\
U_{H_t} &= -H_t^{\sigma_l} (C_t - \chi C_{t-1}/(1+g_t))^{-\sigma_c} \exp\left(\frac{(\sigma_c - 1) H_t^{1+\sigma_l}}{1+\sigma_l}\right) \\
\lambda_t &= \mathbb{E}_t[\beta_{g,t+1}\, R_{t+1}]\ RPS_t\, \lambda_{t+1} \\
\lambda_t &= U_{C,t} - \chi\, \mathbb{E}_t[\beta_{g,t+1} U_{C,t+1}] \\
\frac{-U_{H_t}}{\lambda_t} &= W_{h,t} \\
R_t &= \frac{R_{n,t-1}}{\Pi_t} \\
Y_t &= \frac{Y_t^W - F}{\Delta_t^p} \\
H_{d,t} &= \frac{H_t}{\Delta_t^w} \\
Y_t^W &= (H_{d,t}\, A_t)^\alpha \left(\frac{K_{t-1}}{1+g_t}\right)^{1-\alpha} \\
R_t^K &= \frac{\left(\frac{Y_t^W\,(1-\alpha)\, MC_t}{\frac{K_{t-1}}{1+g_t}} + (1-\delta)\ Q_t\right)}{Q_{t-1}} \\
\Lambda_{t-1,t} &= \frac{\beta_{g,t}\, \lambda_t}{\lambda_{t-1}} \\
1 &= Q_t\ \left(1 - S_t - X_t\, S_t'\right) + \mathbb{E}_t[\Lambda_{t,t+1}\, Q_{t+1}\, S_{t+1}'\ (X_{t+1})^2] \\
\frac{\alpha\, MC_t\, Y_t^W}{H_t} &= W_t \\
MC_t &= \frac{P_t^W}{P_t} \\
K_t &= \left((1-S_t)\ I_t + \frac{K_{t-1}\ (1-\delta)}{1+g_t}\right) \\
X_t &= \frac{(1+g_t)\ I_t}{I_{t-1}}
\end{aligned}
$$

$$
\begin{aligned}
S_t &= \phi_X \,(X_t - 1 - g)^2 \\
S_t' &= 2\,\phi_X\,(X_t - 1 - g) \\
1 &= \Lambda_{t,t+1}\, R^K_{t+1} = 1 \\
Y_t &= C_t + I_t + G_t \\
Y_t &= JJ^p_t - \mathbb{E}_t[(1+g_{t+1})\;\Lambda_{t,t+1}\,\tilde{\Pi}^{\zeta-1}_{t+1}\,JJ^p_{t+1}] \\
\frac{\zeta}{\zeta-1}\,Y_t\,MC_t\,MCS_t &= J^p_t - \mathbb{E}_t\left[(1+g_{t+1})\;\Lambda_{t,t+1}\,\tilde{\Pi}^{\zeta}_{t+1}\,J^p_{t+1}\right] \\
\Lambda_{t,t+1} &= \frac{\beta_{g,t+1} U_{C,t+1}}{U_{C,t}} \\
\tilde{\Pi}_t &= \frac{\Pi_t}{\Pi^{\gamma_p}_{t-1}} \\
P^O_t &= \frac{J^p_t}{JJ^p_t} \\
1 &= \tilde{\Pi}^{\zeta-1}_t + \left(1-\xi_p\right)\,(P^O_t)^{1-\zeta} \\
\Delta^p_t &= \xi_p\,\tilde{\Pi}^{\zeta}_t\,\Delta^p_{t-1} + \left(1-\xi_p\right)\,(P^O_t)^{(-\zeta)} \\
\Pi^w_t &= \Pi_t \frac{W_t(1+g_t)}{W_{t-1}} \\
\tilde{\Pi}^w_t &= \frac{\Pi_t}{\Pi^{\gamma_w}_{t-1}} \\
H_t &= JJ^w_t - \mathbb{E}_t\left[\frac{\Lambda_{t,t+1}\,\xi_w\,(\tilde{\Pi}^w_{t+1})^{\mu_w}}{\tilde{\Pi}_{t+1}(\gamma_w)}\,JJ^w_{t+1}\right] \\
\frac{\mu_w}{\mu_w-1}\,W_{h,t}\,H_t\,MRSS_t &= J^w_t - \mathbb{E}_t[(1+g_{t+1})\Lambda_{t,t+1}\,\xi_w\,(\tilde{\Pi}^w_{t+1})^{\mu_w}\,J^w_{t+1}] \\
W^O_t &= \frac{J^w_t}{W_t\,JJ^w_t} \\
1 &= \xi_w\,\left(\frac{\Pi^w_t\,\tilde{\Pi}_t(\gamma_w)}{\Pi_t}\right)^{\mu_w-1} + (1-\xi_w)\,(W^O_t)^{1-\mu_w} \\
\Delta^w_t &= \xi_w\,(\tilde{\Pi}^w_t)^{\mu_w}\,\Delta^w_{t-1} + (1-\xi_w)\,(W^O_t)^{-\mu_w} \\
Invmarkup_t &= \frac{W_{h,t}}{W_t} \\
\log\left(\frac{R_{n,t}}{\bar{Rn}}\right) &= \rho_r\,\log\left(\frac{R_{n,t-1}}{\bar{Rn}}\right) + (1-\rho_r)\,\left(\theta_\pi\,\log\left(\frac{\Pi_t}{\bar{\Pi}}\right)\right. \\
&+ \left.\theta_y\,\log\left(\frac{Y_t}{\bar{Y}}\right) + \theta_{dy}\,\log\left(\frac{Y_t}{Y_{t-1}}\right)\right) + \log(MPS_t)
\end{aligned}
$$

with AR(1) processes for A_t, G_t, MC_t, $MRSS_t$, IS_t, MPS_t and RPS_t.

2.9.2 Balanced Growth Steady State

With non-zero steady state growth, the steady state for the rest of the system is the same as the zero-growth RBC model except for the following relationships: for particular steady state inflation rate $\Pi_p = \Pi_w > 1$ the NK features of the blanced growth steady state become:

$$\begin{aligned}
R_n &= \Pi R \\
\tilde{\Pi}_p(\gamma) &\equiv \Pi^{1-\gamma} \\
\frac{P^O}{P} = \frac{J^p}{JJ^p} &= \left(\frac{1-\xi_p \tilde{\Pi}_p(\gamma_p)^{\zeta-1}}{1-\xi_p}\right)^{\frac{1}{1-\zeta}} \\
MC = \frac{P^W}{P} &= \left(1-\frac{1}{\zeta}\right)\frac{J^p(1-\beta(1+g)\xi_p\tilde{\Pi}_p(\gamma_p)^{\zeta})}{H_p(1-\beta(1+g)\xi_p\tilde{\Pi}_p(\gamma_p)^{\zeta-1})} \\
&= \text{Inverse of price mark-up} \\
\Delta_p &= \frac{1-\xi_p}{1-\xi_p\tilde{\Pi}_p(\gamma_p)^{\zeta}}\left(\frac{J^p}{JJ^p}\right)^{-\zeta}
\end{aligned}$$

and for wage dynamics

$$\begin{aligned}
\frac{W^O}{W} = \frac{\frac{J^w}{JJ^w}}{\frac{W}{P}} &= \left(\frac{1-\xi_w\tilde{\Pi}_p(\gamma_w)^{\mu-1}}{1-\xi_w}\right)^{\frac{1}{1-\mu}} \\
\frac{J^w}{JJ^w} &= MS_w\frac{W_h}{P}\frac{(1-\beta\xi_w(1+g)\Pi\tilde{\Pi}_p(\gamma_w)^{\mu-1}}{(1-\beta\xi_w\tilde{\Pi}_p(\gamma_w)^{\mu})} \\
\text{i.e., } \frac{\frac{W_h}{P}}{\frac{W}{P}} &= \left(1-\frac{1}{\mu}\right)\frac{\frac{J^w}{JJ^w}}{\frac{W}{P}}\frac{(1-\beta\xi_w\Pi\tilde{\Pi}_p(\gamma_w)^{\mu}}{(1-\beta\xi_w(1+g)\tilde{\Pi}_p(\gamma_w)^{\mu-1})} \\
&= \text{Inverse of wage mark-up} \\
\Delta_w &= \frac{1-\xi_w}{1-\xi_w\tilde{\Pi}_p(\gamma_w)^{\mu}}\left(\frac{\frac{J^w}{JJ^w}}{\frac{W}{P}}\right)^{-\mu}
\end{aligned}$$

2.10 CALIBRATION

In a quarterly model we set steady state values of inflation, hours and growth as $\bar{\Pi} = 1.01$, $\bar{H} = 0.33$ and $\bar{g} = 0.005$. Standard values for the labour share, price elasticity, the discount and depreciation rates and the the inverse of the intertemporal elasticity of substitution are $\alpha = 0.7$, $\zeta = 7$, $\mu = 3$, $\beta = 0.99$, $\delta = 0.025$ and $\sigma_c = 2$. Government spending as a proportion of GDP is set at $g_y = 0.2$. Calvo parameters, indexing, habit and investment adjustment parameters drive the dynamics and should be estimated, but here we choose the following fairly arbitrary values, $\xi_p = \xi_w = 0.75$, $\gamma_p = \gamma_w = 0.5$, $\chi = 0.7$ and $\phi_X = 2$.

Some remaining parameters to be set the persistence parameters in the AR(1) shock processes and the standard deviations of the shocks. Again these are later estimated by Bayesian methods, but for moment persistence parameters are set at 0.75 and shocks have a standard deviation of 1%.

Estimations of the elasticity of labour supply found using microeconomic data depend on factors such as gender, age, marital status and dependants. Pencavel (1986) and Keane (2011) offer surveys of labour supply; restricting the samples to men, the former finds estimates of the elasticity range from 0 to 0.5, with a median 0.2, and the latter finding a larger range of between 0 to 0.7 with an average of 0.31. The elasticities vary significantly due to the listed factors; Reichling and Whalen (2017) give a thorough review of the estimates found in the literature based on microeconomic data, finding that estimates typically range from 0 to over 1. The higher estimates corresponding to married women with children, whereas the labour supply of men is far lower. Combining the results, Reichling and Whalen (2017) propose a range of between 0.27 and 0.53, with a central point estimate of 0.4.

In the real business cycle literature, the response of hours to productivity shocks is a key propagation mechanism and, in the standard model, considered to be adjusted on the intensive margin. The elasticity of labour supply is usually chosen to target the second moment of hours in model simulations and typically calibrated within a range between 2 and 4 (see e.g. Cho and Cooley (1994), King and Rebelo (1999), Chetty *et al.* (2012)), corresponding to a range in δ of between 0.25 and 0.5. The New Keynesian DSGE literature estimate the elasticity within a range that is higher than the microeconomic estimations, although closer to the than the RBC calibrations; for example, Smets and Wouters (2007b) estimate the elasticity to 1.9 (δ = 0.53).

There have been several attempts to explain these differences. Peterman (2016) argues that macroeconomic estimations implicitly include adjustments on both the extensive and intensive margins for the whole population, whereas microeconomic studies are usually focused on the intensive margin, and including these factors can increase the Frisch elasticity to between 2.9 and 3.1. Rogerson and Wallenius (2009) further argue that human capital formation is implicitly included in macroeconomic estimations and can also help reconcile the differences (see also Chetty *et al.* 2012). Hall (2009) proposes a macroeconomic model with labour supply adjustments on the intensive margin and extensive margin via search and match frictions, that reproduces simulations consistent with empirical time series, with a Frisch elasticity of only 0.7 ($\delta = 1.43$).

2.11 DYNARE CODE

- **NK_SW.mod** with steady state files **NK_SW_steadystate.mod** which calls function **ss_fun_NK_SW**. Very similar to Smets and Wouters (2007a)
- There are **no flexi-price, flexi-wage blocs** in these codes and therefore they are only set up for an **implementable rule**.
- But see policy section with a flexi bloc that can have a **conventional Taylor rule**
- **graphs_irfs_compare_NK** Graph plotter for irfs

2.12 EXERCISES

1. Use the options in the code **NK_SW.mod** to explore the effect of introducing wage stickiness into the NK model. Compare impulse responses, volatility, co-movement and the contribution of shocks to the output variance for models with and without wage flexibility.
2. Do the same for external and internal habit.
3. Do the same for models with and without indexation.

3 THE GERTLER-KIYOTAKI MODEL (GK)

The financial market friction in this model is driven by the costs of enforcing contracts (as opposed to private information in the BGG model to come). Financial frictions affect real activity via the impact of funds available to bank but there is no friction in transferring funds between banks and nonfinancial firms. Given a certain deposit level a bank can lend frictionlessly to nonfinancial firms against their future profits. In this regard, firms offer to banks a perfect state-contingent security.

3.1 THE MODEL

Our modelling strategy is to replace the arbitrage condition in the benchmark NK model (2.52), with a banking sector that introduces a wedge between the expected ex ante cost of loans from households, R_t and the return on capital R_t^K. The model closely follows Gertler and Kiyotaki (2010) - henceforth GK - but embeds in our previous NK model with sticky prices in a similar fashion to Gertler and Karadi (2011). Apart from the arbitrage condition, the details of the model are unchanged, so we concentrate on the banking model.

The activity of the bank can be summarized in two phases. In the first one banks raises deposits and equity from the households. In the second phase banks uses the deposits to make loans to firms.

In particular, we have the following sequence of events:

1. Banks raise deposits, d_t from households at a real deposit net rate R_{t+1} over the interval $[t, t+1]$, the 'time period t'.

2. Banks make loans s_t at a price Q_t to firms.

3. The asset against which the loans are obtained is end-of-period capital K_t. Capital depreciates at a rate δ in each period.

The level of the loans depends on the level of the deposits and the net worth of the intermediary. This implies a banking sector's balance sheet of the form:

$$Q_t s_t = n_t + d_t \tag{3.1}$$

Therefore $Q_t s_t$ are the assets of the bank. The liabilities of the bank are household deposits d_t and net worth n_t, the bank capital of its owners, the households.

Net worth of the bank accumulates in stationarized form according to:

$$(1 + g_t)n_t = R_t^K Q_{t-1}s_{t-1} - R_t d_{t-1} \tag{3.2}$$

where (as in the NK model) real returns on bank assets are given by

$$R_t^K = \frac{[r_t^K + (1-\delta)Q_t]}{Q_{t-1}}$$

r_t^K is the gross return (marginal product) of capital and where $r_t^K + (1-\delta)Q_t$ represents the net return after depreciation and the capital quality shock is left out for now.

Banks exit with probability $1 - \sigma_B$ per period and therefore survive for $i-1$ periods and exit in the ith period with probability $(1-\sigma_B)\sigma_B^{i-1}$. Given the fact that bank pays dividends only when it exists, the banker's objective is to maximize expected discounted terminal wealth

$$V_t = \mathbb{E}_t \sum_{i=1}^{\infty} (1-\sigma_B)\sigma_B^{i-1}\Lambda_{t,t+i}n_{t+i} \tag{3.3}$$

where $\Lambda_{t,t+i} = \beta^i \frac{\Lambda_{C,t+i}}{\Lambda_{C,t}}$ is the stochastic discount factor, subject to an incentive constraint for lenders (households) to be willing to supply funds to the banker.

To understand this dynamic problem better we can substitute for d_t from (3.1) and rewrite (3.2) as

$$(1 + g_t)n_t = R_t n_{t-1} + (R_t^K - R_t)Q_{t-1}s_{t-1} \tag{3.4}$$

which says that net worth at the end of period t equals the gross return at the real riskless rate plus the excess return over the latter on the assets. With these returns and Q_t exogenous to the bank, given n_{t-1} at the beginning of period t, net worth in period t is determined by the bank's choice of $\{s_{t+i}\}$, subject to a borrowing constraint.

To motivate an endogenous constraint on the bank's ability to obtain funds, we introduce the following simple agency problem. We assume that after a bank obtains funds, the bank's manager may transfer a fraction of assets to her family. In recognition of this possibility, households limit the funds they lend to banks.

Divertable assets consists of total gross assets $Q_t s_t$. If a bank diverts assets for its personal gain, it defaults on its debt and shuts down. The creditors may re-claim the remaining fraction $1 - \Theta$ of funds. Because its creditors recognize the bank's incentive to divert funds, they will restrict the amount they lend. In this way a borrowing constraint may arise. In order to ensure that bankers do not divert funds the following incentive constraint must therefore hold:

$$V_t \geq \Theta Q_t s_t \tag{3.5}$$

The incentive constraint states that for households to be willing to supply funds to a bank, the bank's franchise value V_t must be at least as large as its gain from diverting funds.

The optimization problem for the bank is to choose a path for loans, $\{s_{t+i}\}$ to maximize V_t subject to (3.1) and (3.2) (or equivalently (3.4)) and (3.5).

3.2 SOLUTION OF THE BANKER'S PROBLEM

The original version of the solution to this appears in GK, but we relegate this to Appendix 3, as the solution presented here is more straightforward and requires less notation. The solution is assumed to take the form

$$V_t = \mathbb{E}_t[\Lambda_{t,t+1}\Omega_{t+1}n_{t+1}] \tag{3.6}$$

where Ω_t is shadow value of a unit of net worth. We write the Bellman equation corresponding to (3.3) as

$$\begin{aligned} V_{t-1} &= \max_{s_t} \mathbb{E}_{t-1}\Lambda_{t-1,t}\left[(1-\sigma_B)n_t + \sigma_B V_t\right] \qquad (3.7) \\ &= \max_{s_t} \mathbb{E}_{t-1}\Lambda_{t-1,t}\left[(1-\sigma_B)n_t + \sigma_B \mathbb{E}_t\Lambda_{t,t+1}\Omega_{t+1}(R_{t+1}n_t + (R^K_{t+1} - R_{t+1})Q_t s_t)\right] \end{aligned}$$

This is subject to the condition that $V_t \geq \Theta Q_t s_t$, which implies the constraint

$$\mathbb{E}_t\Lambda_{t,t+1}\Omega_{t+1}[R_{t+1}n_t + (R^K_{t+1} - R_{t+1})Q_t s_t] \geq \Theta Q_t s_t \tag{3.8}$$

If $\Omega_t \mathbb{E}_t\Lambda_{t,t+1}[R^K_{t+1} - R_{t+1}] \geq \Theta$, then the constraint does not bind and leverage $Q_t s_t / n_t$ is indeterminate. If $\Omega_t \mathbb{E}_t\Lambda_{t,t+1}[R^K_{t+1} - R_{t+1}] < \Theta$, then maximization takes place if and only if the constraint binds, so that the solution is

$$Q_t s_t = \frac{\mathbb{E}_t[\Lambda_{t,t+1}\Omega_{t+1}R_{t+1}]}{\Theta - \Omega_t \mathbb{E}_t[\Lambda_{t,t+1}(R^K_{t+1} - R_{t+1})]} n_t \tag{3.9}$$

It follows that

$$\mu_t = 1 - \sigma_B + \frac{\sigma_B \Theta \mathbb{E}_t[\Lambda_{t,t+1}\Omega_{t+1}R_{t+1}]}{\Theta - \Omega_t \mathbb{E}_t \Lambda_{t,t+1}[R^K_{t+1} - R_{t+1}]} \tag{3.10}$$

Equivalently, defining $\phi_t = Q_t s_t / n_t$, we can rewrite this last equation as

$$\Omega_t = 1 - \sigma_B + \sigma_B \Theta \phi_t \tag{3.11}$$

as in Appendix 3.

Thus we determine ϕ_t in terms of the interest rate wedge $R^K_t - R_t$. Note that in the absence of a binding IC constraint, $\Omega_t = 1$, and $\mathbb{E}_t \Lambda_{t,t+1} R_{t+1} = \mathbb{E}_t \Lambda_{t,t+1} R^K_{t+1} = 1$, which is the arbitrage condition for our NK model, so leverage is indeterminate.

3.3 AGGREGATION

At the aggregate level the banking sector balance sheet is:

$$Q_t S_t = N_t + D_t$$

At the aggregate level net worth is the sum of existing (old) bankers and new bankers:

$$N_t = N_{o,t} + N_{n,t}$$

Net worth of existing bankers equals earnings on assets held in the previous period net cost of deposit finance, multiplied by a fraction σ_B, the probability that they survive until the current period:

$$(1 + g_t)N_{o,t} = \sigma_B\{(r^K_t + (1-\delta)Q_t)S_{t-1} - R_t D_{t-1}\}$$

Since new bankers cannot operate without any net worth, we assume that the family transfers to each one the fraction $\xi_B/(1-\sigma_B)$ of the total value assets of exiting bankers. This implies:

$$(1 + g_t)N_{n,t} = \xi_B[r^K_t + (1-\delta)Q_t]S_{t-1} \tag{3.12}$$

and in aggregate leverage is given by

$$\phi_t = \frac{Q_t S_t}{N_t} \tag{3.13}$$

3.4 SUMMARY OF THE AGGREGATE GK BANKING MODEL

The complete model is the NK model plus the banking sector It is derived above with the NK arbitrage demand for capital relationship replaced with the following equations that represent the banking sector.[1] Here we focus on the case where the borrowing constraint always binds. Then in stationarized form we have

$$\begin{aligned}
Q_t S_t &= \phi_t N_t \\
\phi_t &= \frac{\mathbb{E}_t[\Lambda_{t,t+1}\Omega_{t+1}R_{t+1}]}{\Theta - \Omega_{t+1}\mathbb{E}_t\Lambda_{t,t+1}[R^K_{t+1} - R_{t+1}]} \\
\Omega_t &= 1 - \sigma_B + \sigma_B \Theta \phi_t \\
N_t(1+g_t) &= R^K_t(\sigma_B + \xi_B)Q_{t-1}S_{t-1} - \sigma_B R_t D_{t-1} \\
D_t &= Q_t S_t - N_t
\end{aligned}$$

The link with the NK model is provided by

$$\begin{aligned}
S_t &= K_t \\
R^K_t &= \frac{r^K_t + (1-\delta)Q_t}{Q_{t-1}} \\
r^K_t &\equiv \frac{(1-\alpha)P^W_t Y^W_t (1+g_t)}{K_{t-1}}
\end{aligned}$$

The complete model is the NK model plus the banking sector is illustrated in Figure 3.1.[2]

3.5 DETERMINISTIC STEADY STATE OF THE GK MODEL

The main difference with respect to the basic NK code is that in this model we use a Matlab solver, fsolve, to solve the capital stock. The balanced-growth steady state of the banking sector with the constraint binding is:

$$\begin{aligned}
S &= K \\
Q &= 1
\end{aligned}$$

1 Note that the foc defining the constraint λ_t is superfluous in the system of equations.

2 There is a slight notation clash with $S(X_t)$ denoting investment adjustment costs and S_t bank assets. In the code we refer to the latter as Sasset.

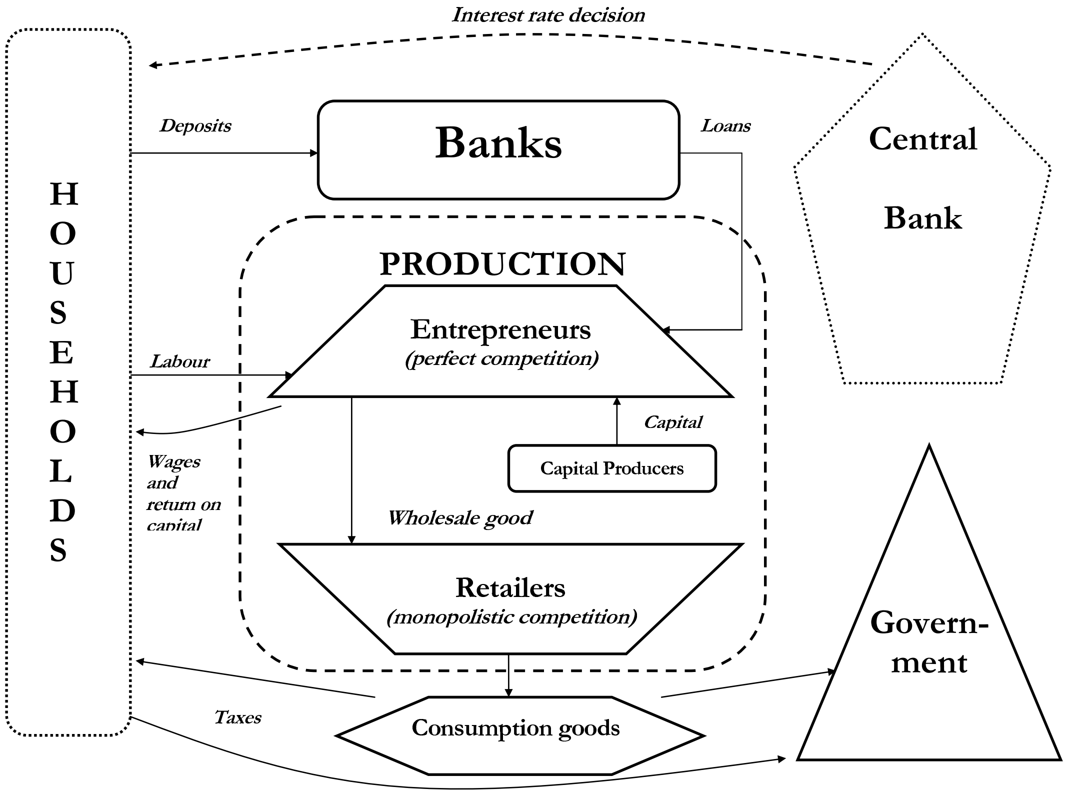


Figure 3.1: **A Model with a Banking Sector**

$$
\begin{aligned}
R &= \frac{R_n}{\Pi} = \frac{1}{\beta_g} \\
\Lambda &= \frac{1}{R} \\
QS &= \phi N \\
\phi &= \frac{\Omega}{\Theta - \Omega(R^K/R - 1)} \\
N &= \frac{R^K(\sigma_B + \xi_B)QS - \sigma_B R}{1 + g - \sigma_B R} \\
D &= QS - N \\
\Omega &= 1 - \sigma_B + \Theta\sigma_B\phi \\
r^K &= \frac{(1-\alpha)P^W Y^W(1+g)}{K} \\
R^K &= \frac{r^K + (1-\delta)Q}{Q}
\end{aligned}
$$

It is very easy to rewrite these equations in recursive form given $\frac{K}{Y^W}$. The external steady state function then uses fsolve in Matlab to solve for $\frac{K}{Y^W}$ and to calibrate three parameters: ϱ, Θ and ξ_B.

3.6 CAPITAL QUALITY SHOCKS

Following the macro-finance literature, in all our banking and NK models we add a capital quality shock, say KQ_{t+1}, that wipes out or enhances capital available in period t going into period $t+1$. $S_t = [(1-\delta)K_{t-1} + (1 - S(X_t))I_t]$ is now 'capital in process' which is transformed by the production process into capital for next period's production according to $K_t = KQ_{t+1}S_t$. Thus capital in process evolves according to

$$S_t = (1-\delta)KQ_t S_{t-1} + (1 - S(X_t))I_t \tag{3.14}$$

Capital quality shock also affects the balance sheet of the banks. Now net returns are given by

$$R_t^K = KQ_t \frac{r_t^K + (1-\delta)Q_t}{Q_{t-1}}$$

It follows from (3.14) and $K_t = KQ_{t+1}S_t$ that

$$K_t = KQ_{t+1}((1-\delta)K_{t-1} + (1 - S(X_t))I_t)$$

3.7 THE GK MODEL WITH OUTSIDE EQUITY

The model, which closely follows Gertler *et al.* (2012) - henceforth GKQ - adds an extra ingredient, the option to raise funds by issuing " outside equity" as well as household deposits.

3.7.1 The Model

Banks raise deposits and equity from the households. In the second phase banks uses the deposits to make loans to firms.

In particular, we have the following sequence of events: The activity of the bank can again be summarized in two phases. In the first one

1. Banks raise deposits, d_t, and outside equity, e_t, from households at a real deposit net rate R_{t+1} and equity net rate R^E_{t+1} respectively over the interval $[t, t+1]$, the 'time period t'.

2. Banks make loans to firms.

3. Loans are s_t at a price Q_t. The asset against which the loans are obtained is end-of-period capital K_t. Capital depreciates at a rate δ in each period. The price of outside equity is $q_t > Q_t$ in our model with financial constraints.

The level of the loans depends on the level of the deposits, value of equity and the net worth of the intermediary. This implies a banking sector's balance sheet of the form:[3]

$$Q_t s_t = n_t + q_t e_t + d_t \tag{3.15}$$

where s_t are claims on non-financial firms to finance capital acquired at the end of period t for use in period $t+1$ and Q_t is the price of a unit of capital so that the assets of the bank. Therefore $Q_t s_t$ are the assets of the bank. The liabilities of the bank are household deposits d_t and net worth n_t.

Net worth of the bank accumulates according to:

$$n_t = R^K_t Q_{t-1} s_{t-1} - R_t d_{t-1} - R^E_t q_{t-1} e_{t-1} \tag{3.16}$$

where real returns on bank assets and equity are given by

$$\begin{aligned} R^K_t &= \frac{[r^K_t + (1-\delta)Q_t]}{Q_{t-1}} \\ R^E_t &= \frac{[r^K_t + (1-\delta)q_t]}{q_{t-1}} \end{aligned}$$

r^K_t is the gross return (marginal product) of capital and where $r^K_t + (1-\delta)Q_t$ represents the net return after depreciation. Again capital quality shocks are omitted at this stage.

As before the banker's objective is to maximize expected discounted terminal wealth

$$V_t = \mathbb{E}_t \sum_{i=1}^{\infty} (1-\sigma_B)\sigma_B^{i-1} \Lambda_{t,t+i} n_{t+i} \tag{3.17}$$

3 In a slight departure from notation elsewhere, lower case denotes the representative bank. Upper case variables later denote aggregates

subject to an incentive constraint for lenders (households) to be willing to supply funds to the banker.

The borrowing constraint is now

$$V_t \geq \Theta(x_t) Q_t s_t \tag{3.18}$$

where $x_t \equiv \frac{q_t e_t}{Q_t s_t}$ is the fraction of bank assets financed by outside equity, $\Theta_t' >$ $, \Theta_t'' > 0$ captures the idea that it is easier to divert assets funded by outside equity than by households.[4] As before, the incentive constraint states that for households to be willing to supply funds to a bank, the bank's franchise value V_t must be at least as large as its gain from diverting funds.

3.7.2 Solution of the Banker's Problem

As before, write the Bellman equation as

$$\begin{aligned} V_{t-1} &= \max_{s_t, e_t} \mathbb{E}_{t-1} \Lambda_{t-1,t} [(1-\sigma_B) n_t + \sigma_B V_t] \\ &= \max_{s_t, e_t} \mathbb{E}_{t-1} \Lambda_{t-1,t} [(1-\sigma_B) n_t + \sigma_B \mathbb{E}_t (\Lambda_{t,t+1} \Omega_{t+1} n_{t+1})] \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} \mathbb{E}_t(\Lambda_{t,t+1}\Omega_{t+1} n_{t+1}) &= \mathbb{E}_t[\Lambda_{t,t+1}\Omega_{t+1}(R_{t+1} n_t + (R_{t+1}^K - R_{t+1}) Q_t s_t + (R_{t+1} - R_{t+1}^E) q_t e_t)] \\ &= \mathbb{E}_t[\Lambda_{t,t+1}\Omega_{t+1}(R_{t+1} n_t + (R_{t+1}^K - R_{t+1} + (R_{t+1} - R_{t+1}^E) x_t) Q_t s_t)] \end{aligned}$$

This is subject to the condition that $V_t \geq \Theta(x_t) Q_t s_t$, which implies the constraint

$$\Omega_t \mathbb{E}_t[\Lambda_{t,t+1}\Omega_{t+1}(R_{t+1} n_t + (R_{t+1}^K - R_{t+1} + (R_{t+1} - R_{t+1}^E) x_t) Q_t s_t)] \geq \Theta(x_t) Q_t s_t \tag{3.20}$$

If the constraint always binds, then the solution from the constraint is

$$Q_t s_t = \frac{\mathbb{E}_t[\Lambda_{t,t+1}\Omega_{t+1} R_{t+1}]}{\Theta(x_t) - \Omega_t \mathbb{E}_t[\Lambda_{t,t+1}\Omega_{t+1}(R_{t+1}^K - R_{t+1} + (R_{t+1} - R_{t+1}^E) x_t)]} n_t \equiv \phi_t n_t \tag{3.21}$$

4 As GKQ explain: " we assume that the fraction of funds the bank may divert depends on the composition of its liabilities. In particular, we assume that at the margin it is more difficult to divert assets funded by short term deposits than by outside equity. Short term deposits require the bank to continuously meet a non-contingent payment. Dividend payments, in contrast, are tied to the performance of the bank's assets, which is difficult for outsiders to monitor. By giving banks less discretion over payouts, short term deposits offer more discipline over bank managers than does outside equity."

subject to the first order conditions from differentiating with respect to s_t, x_t respectively, and using the Lagrange multiplier λ_t for the constraint:

$$(1+\lambda_t)\mathbb{E}_t(\Lambda_{t,t+1}\Omega_{t+1}(R^K_{t+1} - R_{t+1} + (R_{t+1} - R^E_{t+1})x_t)) = \lambda_t\Theta(x_t) \quad (3.22)$$

$$(1+\lambda_t)\mathbb{E}_t\Lambda_{t,t+1}\Omega_{t+1}(R_{t+1} - R^E_{t+1}) = \lambda_t\Theta'(x_t) \quad (3.23)$$

By eliminating the Lagrange multiplier, this reduces to:

$$\mathbb{E}_t(\Lambda_{t,t+1}\Omega_{t+1}(R^K_{t+1} - R_{t+1} + (R_{t+1} - R^E_{t+1})x_t)) = \mathbb{E}_t\Lambda_{t,t+1}\Omega_{t+1}(R_{t+1} - R^E_{t+1})\frac{\Theta(x_t)}{\Theta'(x_t)} \quad (3.24)$$

Given the definition of ϕ_t in (3.21), we obtain, similarly to the internal equity case:

$$\Omega_t = 1 - \sigma_B + \sigma_B\Theta(x_t)\phi_t \quad (3.25)$$

with Ω_t corresponding to the GKQ solution in Appendix 3.

3.7.3 Aggregation

At the aggregate level the banking sector balance sheet is:

$$Q_tS_t = N_t + q_t + D_t$$

At the aggregate level net worth is the sum of existing (old) bankers and new bankers:

$$N_t = N_{o,t} + N_{n,t}$$

Net worth of existing bankers equals earnings on assets held in the previous period net cost of deposit finance, multiplied by a fraction σ_B, the probability that they survive until the current period:

$$(1+g_t)N_{o,t} = \sigma_B\{(r^K_t + (1-\delta)Q_t)S_{t-1} - (r^K_t + (1-\delta)q_t)E_{t-1} - R_tD_{t-1}\}$$

Since new bankers cannot operate without any net worth, we assume that the family transfers to each one the fraction $\xi_B/(1-\sigma_B)$ of the total value assets of exiting bankers. This implies:

$$(1+g_t)N_{n,t} = \xi_B[r^K_t + (1-\delta)Q_t]S_{t-1} \quad (3.26)$$

In aggregate leverage (in terms of net worth) is given by

$$\phi_t = \frac{Q_tS_t}{N_t} \quad (3.27)$$

Finally in the absence of credit policy by the authorities the model is closed by household arbitrage conditions

$$\mathbb{E}_t[\Lambda_{t,t+1}R_{t+1}] = \mathbb{E}_t[\Lambda_{t,t+1}R^E_{t+1}] = 1 \tag{3.28}$$

where

$$R^E_t = \frac{r^K_t + (1-\delta)q_t}{q_{t-1}} \tag{3.29}$$

3.7.4 Summary of the Aggregate GK Model with Outside Equity

We again focus on the case where the borrowing constraint always binds. Then in stationarized form we have

$$\begin{aligned}
S_t &= K_t \\
x_t &= \frac{q_t E}{Q_t S_t} \\
\frac{\Theta(x_t)}{\Theta'(x_t)} &= \frac{\mathbb{E}_t(\Lambda_{t,t+1}\Omega_{t+1}(R^K_{t+1} - R_{t+1} + (R_{t+1} - R^E_{t+1})x_t))}{\mathbb{E}_t\Lambda_{t,t+1}\Omega_{t+1}(R_{t+1} - R^E_{t+1})} \\
Q_t S_t &= \phi_t N_t \\
\phi_t &= \frac{\mathbb{E}_t[\Lambda_{t,t+1}\Omega_{t+1}R_{t+1}]}{\Theta(x_t) - \mathbb{E}_t[\Lambda_{t,t+1}\Omega_{t+1}(R^K_{t+1} - R_{t+1} + (R_{t+1} - R^E_{t+1})x_t)]} \\
(1+g_t)N_t &= [r^K_t + (1-\delta)Q_t](\sigma_B + \xi_B)S_{t-1} - \sigma_B(r^K_t + (1-\delta)q_t)E_{t-1} - \sigma_B R_t D_{t-1} \\
D_t &= Q_t S_t - N_t - q_t E \\
\Omega_t &= 1 - \sigma_B + \sigma_B \Theta_t \phi_t \\
\mathbb{E}_t[\Lambda_{t,t+1}R_{t+1}] &= \mathbb{E}_t[\Lambda_{t,t+1}R^E_{t+1}] = 1 \\
R^K_t &= \frac{r^K_t + (1-\delta)Q_t}{Q_{t-1}} \\
R^E_t &= \frac{r^K_t + (1-\delta)q_t}{q_{t-1}} \\
r^K_t &= \frac{(1-\alpha)P^W_t Y^W_t}{K_{t-1}/(1+g_t)} \\
\mathbb{E}_t[\Lambda_{t,t+1}R_{t+1}] &= \mathbb{E}_t[\Lambda_{t,t+1}R^E_{t+1}] = 1
\end{aligned}$$

3.7.5 Deterministic Steady State

The non-zero-growth, non-zero-net-inflation deterministic steady state of the banking sector is:

$$
\begin{aligned}
R^E &= R = \frac{R_n}{\Pi} = \frac{1}{\beta_g} \\
\mu_e &= 0 \\
S &= K \\
Q &= 1 \\
x &= \frac{qE}{QS} \\
\Lambda &= \beta \\
\frac{\Theta'(x)}{\Theta(x)} &= \frac{1 - R^E/R}{R^K/R - 1 + (1 - R^E/R)x} \\
QS &= \phi N \\
\phi &= \frac{\Omega}{\Theta(x) - \Omega(R^K/R - 1 + (1 - R^E/R)x)} \\
N(1+g) &= [r^K + (1-\delta)Q](\sigma_B + \xi_B)S - \sigma_B(r^K + (1-\delta)q)E - \sigma_B RD \\
D &= QS - N - qE \\
\Omega &= 1 - \sigma_B + \Theta_t \sigma_B \phi \\
\Lambda R &= \Lambda R_e = 1 \\
R^K &= \frac{r^K + (1-\delta)Q}{Q} \\
R_e &= \frac{r^K + (1-\delta)q}{q} \\
r^K &= \frac{(1-\alpha)P^W Y^W (1+g)}{K}
\end{aligned}
$$

Again given K it is easy to order these equations recursively.

3.7.6 Calibration and Functional Form

Non-financial parameters and steady state values of Π, g and H are calibrated as in the benchmark NK model. The parameters of the banking sector are calibrated in the following way. Following GK, choose the value of σ_B so that the bankers survive 8 years (32 quarters) on average. Then with our quarterly model, $\frac{1}{1-\sigma_B} = 32$. The values of Θ and ξ_B are computed to hit an economy wide leverage ratio of four and to have an average credit spread of 100 basis points per year. Then in our quarterly model $\sigma_B = 0.9688$, $\phi = 4$ and $R^K - R = 0.0025$.

For the function $\Theta(x_t)$ we choose

$$\Theta_t \equiv \Theta(x_t) = \theta_{FF}(1 + \epsilon x_t + \kappa_{FF} x_t^2/2) \tag{3.30}$$

Since in the steady state $R = R^E$, $\mu_e = \Theta' = 0$. We then obtain

$$x = -\frac{\epsilon}{\kappa_{FF}} \tag{3.31}$$

We set $\epsilon = -2$ (as in GK). Then by choosing a target for x we can pin down the remaining parameter κ. Note that only $\frac{\epsilon}{\kappa_{FF}}$ is pinned down in the deterministic steady state, so ϵ remains undetermined using this calibration strategy.[5]

Parameters θ_{FF}, κ_{FF} and ξ_B are calibrated to hit a total leverage target $\frac{QS}{N+qE} = 4$, a spread $R^K - R = 0.01/4$. We choose a targets for the outside equity ratio, 0.15 giving calibrated values for θ, ξ_B and κ_{FF} shown in Table 3.1.

Parameter	Calibrated Value
θ_{FF}	0.4274
κ_{FF}	13.333
ξ_B	0.0023
ϵ	-2

Table 3.1: GK-equity Model with Internal Habit. Calibrated Parameters.

5 However if we had two scenarios, say a 'low risk' and 'high risk' (as in Gertler *et al.* (2012)) then using the *stochastic* steady state one can in principle pin down this remaining parameter. Alternatively these authors use the *risky steady state* which can be solved analytically.

3.8 DYNARE CODE

- The GK banking models for SW preferences, **GK_SW.mod**, is in folder ***GK***
- External steady state matlab file **GK_SW_steadystate.m** then calls function and **ss_fun_GK_SW.m** respectively. fsolve then solves for the steady state K and performs the calibration.
- A matlab file **graphs_irfs_compare_NK_GK_BGG.m** to compare the irfs of the NK and banking models is also included in the folder.

3.9 EXERCISES

1. Add and capital quality shock KQ_t to the GK model with equity. Use the graph plotter to compare KQ_t and IS_t shocks
2. **Take Home Exercise**:
   - Modify the banking sector of the GK model with SW preferences so that it takes the simpler form explained in Appendix 4 of the Notes.
   - You will need to modify all three files ***GK.mod***, ***GK_steadystate.m*** and ***ss_fun_GK.m***.
   - Confirm that this alternative leads to the same equilibrium

4 THE BGG FINANCIAL ACCELERATOR MODEL

In a 'costly state verification model' due originally to Townsend (1979), the modelling strategy is once again to replace $\mathbb{E}_t[\Lambda_{t,t+1}R_{t+1}] = \mathbb{E}_t[\Lambda_{t,t+1}R^K_{t+1}]$ with a wedge that arises from the friction between a the risk neutral entrepreneur and a financial intermediary. The former borrow from the latter to purchase capital from capital producers at a price Q_t and combine it with labour through a production technology to produce wholesale output. In order to ensure they cannot grow out of the financial constraint, entrepreneurs exit with probability σ_E. As we shall see this setup introduces a wedge between the expected ex post (non-riskless) rate, $\mathbb{E}_t[R_{t+1}]$ and the expected return on capital $\mathbb{E}_t[R^K_{t+1}]$.[1]

4.1 THE MODEL

The entrepreneur seeks loans l_t to bridge the gap between its net worth $n_{E,t}$ and the expenditure on new capital $Q_t k_t$, all end-of-period. Thus

$$l_t = Q_t k_t - n_{E,t} \tag{4.1}$$

where the entrepreneur's *real* net worth accumulates[2] according to

$$n_{E,t} = R^K_t Q_{t-1} k_{t-1} - \frac{R_{l,t-1}}{\Pi_t} l_{t-1}$$

where R^K_t is the *real* return on capital as in the NK model and $R_{l,t}$ is the *nominal* loan rate to be decided in the contract. Note that all variables other than the loan rate are expressed in real terms.

In each period an idiosyncratic capital quality shock, ψ_t results in a return $R^K_t \psi_t$ which is the entrepreneur's private information. Default in period $t+1$

1 The model setup draws upon Faia and Monacelli (2007) as well as Bernanke *et al.* (1999). We are grateful to Jonathan Swarbrick for helping with this section.

2 We stationarize later in Section 4.2.

occurs when net worth becomes negative, i.e., when $n_{E,t+1} < 0$ and shock falls below a threshold $\bar{\psi}_{t+1}$ given by

$$\bar{\psi}_{t+1} = \frac{R_{l,t} l_t}{\Pi_{t+1} R^K_{t+1} Q_t k_t} \tag{4.2}$$

With the idiosyncratic shock, ψ_t drawn from a density $f(\psi_t)$ with a lower bound ψ_{min}, the probability of default is then given by

$$p(\bar{\psi}) = \int_{\psi_{min}}^{\bar{\psi}_{t+1}} f(\psi) d\psi$$

In the event of default the bank receives the assets of the firm and pays a proportion μ of monitoring costs to observe the realized return. Otherwise the bank receives the full payment on its loans, $R_{l,t} l_t / \Pi_{t+1}$ where $R_{l,t}$ is the agreed loan rate at time t.

At the heart of the model is the bank's *incentive compatibility (IC) constraint* given by

$$\mathbb{E}_t \left[(1-\mu) R^K_{t+1} Q_t k_t \int_{\psi_{min}}^{\bar{\psi}_{t+1}} \psi f(\psi) d\psi + (1 - p(\bar{\psi}_{t+1})) \frac{R_{l,t}}{\Pi_{t+1}} l_t \geq R_{t+1} l_t \right] \tag{4.3}$$

The LHS of (4.3) is the expected return to the bank from the contract averaged over all realizations of the shock, the RHS is the return from a riskless bond. To be incentive compatible, the expected return from the contract must be equal or greater than the intermediary's opportunity cost, which is the rate R_{t+1}.

Eliminating the real loan rate from (4.2), the IC constraint becomes

$$\mathbb{E}_t \left[R^K_{t+1} Q_t k_t \left((1-\mu) \int_{\psi_{min}}^{\bar{\psi}_{t+1}} \psi f(\psi) d\psi + \bar{\psi}_{t+1} (1 - p(\bar{\psi}_{t+1})) \right) \geq R_{t+1} l_t \right] \tag{4.4}$$

Now define $\Gamma(\bar{\psi}_{t+1})$ to be the expected fraction of net capital received by the lender (the bank) and $\mu G(\bar{\psi}_{t+1})$ to be expected monitoring costs where

$$\Gamma(\bar{\psi}_{t+1}) \equiv \int_{\psi_{min}}^{\bar{\psi}_{t+1}} \psi f(\psi) d\psi + \bar{\psi}_{t+1} (1 - p(\bar{\psi}_{t+1})) \tag{4.5}$$

$$G(\bar{\psi}_{t+1}) \equiv \int_{\psi_{min}}^{\bar{\psi}_{t+1}} \psi f(\psi) d\psi \tag{4.6}$$

Then the optimal contract for the risk neutral entrepreneur solves

$$\max_{\bar{\psi}_{t+1}, k_t} \mathbb{E}_t \left[(1 - \Gamma(\bar{\psi}_{t+1})) R^K_{t+1} Q_t k_t \right]$$

given initial net worth $n_{E,t}$, subject to the IC constraint (4.4) which, using (4.1), (4.5) and (4.6), can be rewritten as

$$\mathbb{E}_t\left[R^K_{t+1}Q_tk_t\left[\Gamma(\bar{\psi}_{t+1})-\mu G(\bar{\psi}_{t+1})\right]\geq R_{t+1}(Q_tk_t-n_{E,t})\right]$$

Let λ_t be the Lagrange multiplier associated with the IC constraint. Then the first order conditions are

$$\begin{aligned} k_t &: \mathbb{E}_t\left[(1-\Gamma(\bar{\psi}_{t+1})R^K_{t+1}+\lambda_t\left[(\Gamma(\bar{\psi}_{t+1})-\mu G(\bar{\psi}_{t+1}))R^K_{t+1}-R_{t+1}\right]\right]=0 \\ \bar{\psi}_{t+1} &: \mathbb{E}_t\left[-\Gamma'(\bar{\psi}_{t+1})+\lambda_t(\Gamma'(\bar{\psi}_{t+1})-\mu G'(\bar{\psi}_{t+1}))\right]=0 \end{aligned}$$

plus the binding IC condition if $\lambda_t > 0$ with $\lambda_t = 0$ if it does not bind. Combining these two conditions, we arrive at

$$\mathbb{E}_t[R^K_{t+1}]=\mathbb{E}_t[\rho(\bar{\psi}_{t+1})R_{t+1}] \tag{4.7}$$

where the *premium on external finance*, $\rho(\bar{\psi}_{t+1})$ is given by[3]

$$\rho(\bar{\psi}_{t+1})=\frac{\Gamma'(\bar{\psi}_{t+1})}{[(\Gamma(\bar{\psi}_{t+1})-\mu G(\bar{\psi}_{t+1}))\Gamma'(\bar{\psi}_{t+1})+(1-\Gamma(\bar{\psi}_{t+1}))(\Gamma'(\bar{\psi}_{t+1})-\mu G'(\bar{\psi}_{t+1}))]}$$

The equation (4.7) which replaces the no arbitrage condition in the NK model is the crucial result coming out of the BGG model. Notice that in the limiting case as $\bar{\psi}_{t+1}$ and the probability of default tend to zero, and as monitoring costs μ disappear, $\Gamma \to 0$ and the risk premium $\rho(\bar{\psi}_{t+1}) \to 1$ returning us to the arbitrage condition in the NK model with a risk-neutral wholesale firm.

So far we have set out the optimizing decision of the representative entrepreneur. We now aggregate assuming that entrepreneurs exit with fixed probability $1-\sigma_E$. To allow new entrants start up we assume exiting entrepreneurs transfer a proportion ξ_E of their wealth to new entrants[4] Aggregate net worth then accumulates according to

$$N_{E,t}=(\sigma_E+\xi_E)(1-\Gamma(\bar{\psi}_t))R^K_tQ_{t-1}K_{t-1}$$

and on exiting the entrepreneur consumes

$$C_{E,t}=(1-\sigma_E)(1-\xi_E)(1-\Gamma(\bar{\psi}_t))R^K_tQ_{t-1}K_{t-1}$$

3 Note that $\rho(\bar{\psi}_{t+1})\geq 1$ iff $\mu \geq 0$.

4 This is a comparable mechanism to the GK model that follows. BGG assume a different mechanism in which entrepreneurs supplement their income by working in the general labour market.

The resource constraint becomes

$$Y_t = C_t + C_{E,t} + G_t + I_t + \mu G(\bar{\psi}_t))R_t^K Q_{t-1} K_{t-1}$$

The equilibrium is completed with the aggregate IC constraint

$$\mathbb{E}_t \left[R_{t+1}^K Q_t K_t \left[\Gamma(\bar{\psi}_{t+1}) - \mu G(\bar{\psi}_{t+1}) \right] \right] = \mathbb{E}_t \left[R_{t+1}(Q_t K_t - N_{E,t}) \right]$$

which pins down the contract rate $R_{l,t}$. Other post-recursive macroeconomic outcomes of interest are loans and the threshold shock value given by

$$\begin{aligned} L_t &= Q_t K_t - N_{E,t} \\ \bar{\psi}_t &= \frac{R_{l,t-1} L_{t-1}}{R_t^K Q_{t-1} k_{t-1}} \frac{1}{\Pi_t} \\ R_t^K &= \frac{r_t^K + (1-\delta) Q_t}{Q_{t-1}} \\ r_t^K &= \frac{(1-\alpha) P_t^W Y_t^W}{K_{t-1}} \end{aligned}$$

which completes the financial side of the model.

4.2 SUMMARY OF BGG EQUILIBRIUM

We now have arrived at the BGG 'financial accelerator' equilibrium which, now in stationarized, form can be summarized as follows.

$$\begin{aligned} \mathbb{E}_t[R_{t+1}^K] &= \mathbb{E}_t[\rho(\bar{\psi}_{t+1}) R_{t+1}] \\ (1+g_t) N_{E,t} &= (\sigma_E + \xi_E)(1 - \Gamma(\bar{\psi}_t)) R_t^K Q_{t-1} K_{t-1} \\ \mathbb{E}_t \left[R_{t+1}^K Q_t K_t \left[\Gamma(\bar{\psi}_{t+1}) - \mu G(\bar{\psi}_{t+1}) \right] \right] &= \mathbb{E}_t \left[R_{t+1}(Q_t K_t - N_{E,t}) \right] \text{ or} \\ \phi_t \mathbb{E}_t \left[R_{t+1}^K \left[\Gamma(\bar{\psi}_{t+1}) - \mu G(\bar{\psi}_{t+1}) \right] \right] &= (\phi_t - 1) \mathbb{E}_t \left[R_{t+1} \right] \end{aligned}$$

where $\phi_t \equiv \frac{Q_t K_t}{N_{E,t}}$ is the leverage ratio and

$$\rho(\bar{\psi}_{t+1}) = \frac{\Gamma'(\bar{\psi}_{t+1})}{[(\Gamma(\bar{\psi}_{t+1}) - \mu G(\bar{\psi}_{t+1}))\Gamma'(\bar{\psi}_{t+1}) + (1 - \Gamma(\bar{\psi}_{t+1}))(\Gamma'(\bar{\psi}_{t+1}) - \mu G'(\bar{\psi}_{t+1}))]}$$

with a resource constraint:

$$Y_t = C_t + C_{E,t} + G_t + I_t + \mu G(\bar{\psi}_t)) R_t^K Q_{t-1} K_{t-1} / (1 + g_t)$$

$$(1+g_t)C_{E,t} = (1-\sigma_E)(1-\xi_E)(1-\Gamma(\bar{\psi}_t))R_t^K Q_{t-1}K_{t-1}$$

and post-recursive equations

$$\begin{aligned}
L_t &= Q_t K_t - N_{E,t} \\
\bar{\psi}_t &= \frac{R_{l,t-1}L_{t-1}}{R_t^K Q_{t-1}K_{t-1}}\frac{1}{\Pi_t} \\
R_t^K &= \frac{r_t^K + (1-\delta)Q_t}{Q_{t-1}} \\
r_t^K &= \frac{(1-\alpha)P_t^W Y_t^W}{K_{t-1}/(1+g_t)}
\end{aligned}$$

4.3 DETERMINISTIC STEADY STATE

The balanced growth deterministic steady state is given by

$$\begin{aligned}
R^K &= \rho(\bar{\psi})R \\
N_E &= (\sigma_E + \xi_E)(1-\Gamma(\bar{\psi}))R^K QK/(1+g) \\
R^K QK\,[\Gamma(\bar{\psi}) - \mu G(\bar{\psi})] &= R(QK - N_E) \text{ or} \\
R^K[\Gamma(\bar{\psi}) - \mu G(\bar{\psi})] &= R(\phi - 1)
\end{aligned}$$

where $\phi \equiv \frac{QK}{N_E}$ and

$$\rho(\bar{\psi}) = \frac{\Gamma'(\bar{\psi})}{[(\Gamma(\bar{\psi}) - \mu G(\bar{\psi}))\Gamma'(\bar{\psi}) + (1-\Gamma(\bar{\psi}))(\Gamma'(\bar{\psi}) - \mu G'(\bar{\psi}))]}$$

with a resource constraint

$$\begin{aligned}
Y &= C + C_E + G + I + \mu G(\bar{\psi})R^K QK/(1+g) \\
C_E &= (1-\sigma_E)(1-\xi_E)(1-\Gamma(\bar{\psi}))R^K QK/(1+g)
\end{aligned}$$

and post-recursive equations

$$\begin{aligned}
L &= QK - N_E \\
R_l &= \frac{\bar{\psi}R^K QK/(1+g)}{L_t}\Pi \\
Q &= 1 \\
R^K &= \frac{r^K + (1-\delta)Q}{Q} \\
r^K &= \frac{(1-\alpha)P^W Y^W (1+g)}{K}
\end{aligned}$$

4.4 CHOICE OF DENSITY FUNCTION

We choose a *log-normal distribution* for ψ, $\log(\psi) \sim \mathcal{N}\left(-\frac{\sigma_\psi^2}{2}, \sigma_\psi^2\right)$. With the mean set to $-\frac{\sigma_\psi^2}{2}$, put $\mathbb{E}[\psi] = 1$. This which has the benefit of being mean preserving if extending to consider volatility in σ_ψ. We then have

$$
\begin{aligned}
p(\bar{\psi}_t) &= \int_0^{\bar{\psi}_t} f\left(\psi; -\frac{\sigma_\psi^2}{2}, \sigma_\psi^2\right) d\psi \\
G(\bar{\psi}_t) &\equiv \int_0^{\bar{\psi}_t} \psi\, f\left(\psi; -\frac{\sigma_\psi^2}{2}, \sigma_\psi^2\right) d\psi \\
\Gamma(\bar{\psi}_{t+1}) &\equiv G(\bar{\psi}_t) + \bar{\psi}_t(1 - p(\bar{\psi}_t))
\end{aligned}
$$

Then it can be shown that

$$
\begin{aligned}
G'(\bar{\psi}_t) &= \frac{1}{\sigma_\psi\sqrt{2\pi}} \exp\left[-\frac{\left(\log(\bar{\psi}_t) + \frac{1}{2}\sigma_\psi^2\right)^2}{2\sigma_\psi^2}\right] \\
\Gamma'(\bar{\psi}_{t+1}) &= 1 - p(\bar{\psi}_t)
\end{aligned}
$$

Using the first order conditions from the contract solution we then have

$$
\lambda_t = \frac{\mathbb{E}_t\left[1 - p(\bar{\psi}_t)\right]}{\mathbb{E}_t\left[1 - p(\bar{\psi}_t) - \mu G'(\bar{\psi}_{t+1})\right]}
$$

An alternative choice a *uniform distribution* on support $\psi \in [1 - A_\psi, 1 + A_\psi]$; i.e.,

$$
f(\psi) = \begin{cases} \frac{1}{2A_\psi} & \text{if } \psi \in [1 - A_\psi, 1 + A_\psi] \\ 0 & \text{otherwise} \end{cases}
$$

Then $\int_{-\infty}^{\infty} f(\psi) d\psi = 1$ and

$$
\begin{aligned}
p(\psi) &= \frac{1}{2A_\psi}\left(\psi - 1 + A_\psi\right) \\
\Gamma(\psi) &= \frac{1}{4A_\psi}\left(\psi^2 - (1 - A_\psi)^2\right) + \psi(1 - p(\psi)) \\
G(\psi) &= \frac{1}{4A_\psi}\left(\psi^2 - (1 - A_\psi)^2\right)
\end{aligned}
$$

$$\Gamma'(\psi) = \frac{1}{2A_\psi}(1-\psi)+\frac{1}{2}$$
$$G'(\psi) = \frac{\psi}{2A_\psi}$$

4.5 CALIBRATION

The parameter values used in the NK model with SW preferences are as before. Additional financial parameters to calibrate are σ_ψ, σ_E, ξ_E and μ. These four parameters are calibrated to hit four targets: a default probability $p(\bar{\psi}) = 0.02$ (as in Faia and Monacelli (2007)), $\rho(\bar{\psi}) = 1.0025$ corresponding to a credit spread of 100 basis points as in GK, an entrepreneur leverage $\frac{QK}{N_E} = 2$ as in Bernanke *et al.* (1999) (rather lower than the bank leverage of 4 as in GK) and entrepreneurial consumption $\frac{C_E}{Y} = 0.075$. In the external steady state m.file we solve for parameters σ_ψ, σ_E, ξ_E, μ. In addition we solve for two endogenous steady-state variables K, H and $\bar{\psi}$ making eight variables to solve in fsolve. Then the calibrated parameters are given in Table 5.2.[5]

Parameter	Calibrated Value
σ_ψ	0.3135
σ_E	0.9764
ξ_E	0.0067
μ	0.0284

Table 4.1: BGG Model with SW Preferences. Calibrated Parameters

Compared with those in Faia and Monacelli (2007) who set $\mu = 0.25$, we find small monitoring costs, but this calibration is necessary to hit a leverage substantially above unity.

5 In the code we avoid a notational clash with μ already defined as a labour supply elasticity by denoting the monitoring cost parameter above as μ_{FF}.

4.6 DYNARE CODE

The mod file code for the BGG model is **BGG_normal_SW.mod** set up with an external steady state that solves for the calibrated parameters above and for K and $\bar{\psi}$ using fsolve and an external steady state function.

4.7 EXERCISES

1. Compare irfs for the NK, GK and BGG models with SW preferences.
2. In the BGG model now choose a *uniform distribution* for ψ and compare with the normal distribution.

5 OVERVIEW OF CALIBRATION OF BANKING MODELS

We calibrate a number of parameters across the models and estimate the remaining ones. The following tables contain discussions for the different calibrations.

5.1 GK MODEL WITH EQUITY

For the function $\Theta(x_t)$ we choose the following parametric form

$$\Theta_t \equiv \Theta(x_t) = \theta(1 + \epsilon x_t + \kappa x_t^2/2) \tag{5.1}$$

Since in the steady state $R = R_e$, $\mu_e = \Theta' = 0$. We then obtain

$$x = -\frac{\epsilon}{\kappa} \tag{5.2}$$

We set $\epsilon = -2$ (as in GK). Then by choosing a target for x we can pin down the remaining parameter κ. Note that only $\frac{\epsilon}{\kappa}$ is pinned down in the deterministic steady state, so ϵ remains undetermined using this calibration strategy.[1]

We use data on US Banks from 1976-2008 via call reports to calibrate a number of the banking parameters. In particular, we set $\sigma_B = 0.978$ as banks survive on average 77 quarters in our sample. This dataset contains key information regarding all US commercial banks on a quarterly frequency including total assets, commercial and industrial loans, and regulatory capital (outside equity) that we use in the calibration process. We choose a target for the outside equity ratio of 0.1 based on the average capital holdings of banks in the call report data. Moreover, we choose a total leverage target $\frac{QS}{N+qE} = 5$ which corresponds roughly to the total leverage observed in our data. We also set target for the spread $R^K - R = 0.0069$ which corresponds to the difference between seasoned BAA bonds and the federal funds rate.

Using these calibration targets, the parameters θ, κ and ξ_B can be pinned down giving calibrated values for θ, ξ_B and κ shown below.

1 However if we had two scenarios, say a 'low risk' and 'high risk' (as in Gertler *et al.* (2012)) then using the *stochastic* steady state one can in principle pin down this remaining parameter. Alternatively these authors use the *risky steady state* which can be solved analytically.

Parameter	Calibrated Value
σ_B	0.978
ϱ	0.8629
ξ_B	0.0017
θ	0.4269
ϵ	-2
κ	13.333

Table 5.1: GK-equity Model. Parameters to Calibrate

5.2 BGG MODEL

The parameter values used in the NK model are as before.[2] Additional financial parameters to calibrate are A_ψ, σ_E, ξ_E and μ. These four parameters are calibrated to hit four targets: a default probability $p(\bar{\psi}) = 0.02$ (as in Faia and Monacelli (2007)), $\rho(\bar{\psi}) = 1.0025$ corresponding to a credit spread of 100 basis points as in GK, an entrepreneur leverage $\frac{QK}{N_E} = 2$ as in Bernanke *et al.* (1999) (rather lower than the bank leverage of 4 as in GK) and entrepreneurial consumption $\frac{C_E}{Y} = 0.1$. In the external steady state m.file we solve for parameters ϱ, A_ψ, σ_E, ξ_E, μ and endogenous variables K/Y^W and ψ. Then the calibrated parameters are given in Table 5.2.

Parameter	Calibrated Value
A_ψ	0.5218
σ_E	0.9632
ξ_E	0.0172
μ	0.010

Table 5.2: BGG Model. Calibrated Parameters

2 Note that, given the change in the resource constraint of the economy with respect to the NK case the steady state external file now solves for the steady state value of hours given the calibrated value of ϱ.

6 BAYESIAN ESTIMATION OF MODELS

6.1 BAYESIAN METHODOLOGY

Bayesian analysis requires:

- Initial information ⇒ Prior distribution
- Data ⇒ Likelihood density or the probability of observing the data given the model and parameters
- Prior and Likelihood ⇒ Bayes theorem ⇒ Posterior distribution
- Posterior distribution used for confidence intervals for parameters and impulse responses.
- The posterior distribution also provides information regarding identification of parameters - how much information does the data provide on parameters?

Bayesian estimation is a full information systems estimation method (much like ML). It can be thought of as 'hybrid' approach between informal calibration and ML. In the absence of prior information it converges to ML and if we are sure the priors are correct we are back to calibration. It uses prior information to identify key structure parameters enabling the utilization of additional sources of information. A major problem with likelihood approaches is that the likelihood surface can be flat (or almost flat) in some directions. Priors then add 'curvature' to the likelihood. The Bayesian approach allows straightforward facilities for the construction of confidence intervals for parameter estimates and impulse responses, forecasting and model comparison.

In practice, the Bayesian approach uses the log-linear approximation of the original model's non-linear optimality conditions around a non-stochastic steady state, obtaining a linear rational expectations system, which is then solved for the state-space form in its predetermined variables. Subsequently, standard

Kalman recursions are applied to compute the likelihood function[1] which, combined with the prior assumptions about model parameters to be estimated, allows us to evaluate their posterior probability.

Bayesian analysis is based on a few simples rules of probability. First **some notation**: Suppose A, B are random variables (or events), then

$$\begin{aligned} \text{probability of event A} &\equiv p(A) \\ \text{probability of A and B} &\equiv p(A,B) \text{ or } p(A \cap B) \\ \text{probability of A given B} &\equiv p(A|B) = P(A) \text{ if A, B are independent} \end{aligned}$$

Then, by definition of conditional probability

$$p(A|B) \equiv \frac{p(A,B)}{p(B)}$$

Reversing the roles of events A and B , we also have

$$p(B|A) \equiv \frac{p(A,B)}{p(A)}$$

Equating these expressions and rearranging, we arrive at the famous **Bayes' rule**:

$$p(B|A) = \frac{p(A|B)p(B)}{p(A)}$$

In Bayesian econometrics, we want to use data (say T data points $y \equiv (y_1, y_2, \cdots y_T)$) to learn about the model's parameters (say n variables, $\theta = (\theta_1, \theta_2, \cdots \theta_n)$) A Bayesian approach does just that: replacing B by θ and A by y

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

Our focus is on $p(\theta|y)$: given the data, what can we tell about θ? Whereas classical (frequentist) econometrics treats θ as some unknown fixed value(s), Bayesian econometrics assumes that, if θ is unknown, then it should be expressed using rules of probability (i.e., θ is effectively a random object).

Noting that we're interested in θ, we can drop $p(y)$, so

$$p(\theta|y) \propto p(y|\theta)p(\theta)$$

1 The likelihood function is computed under the assumption of normally distributed disturbances.

Whereas ML maximizes $p(y|\theta)$ wrt θ; Bayesian estimation maximizes $p(\theta|y)$. Note that the prior $p(\theta)$ gives the surface more curvature,

The posterior density $p(\theta|y)$ which summarizes what we know about θ after (hence *posterior*) seeing the data. $p(y|\theta)$ is the likelihood density given the model parameters - also denoted as $L(\theta;y)$. $p(\theta)$ is the prior density $p(\theta|y) \propto p(y|\theta)p(\theta)$ is like an updating rule: the data allow us to update our priors about θ, resulting in the posterior, which combines data and non-data information. The likelihood density is computed using the *Kalman Filter* which is a recursive forecasting procedure for the unobserved states given the observables in the linear state space form (see Miao, chapters 10 and 15).

6.2 COMPUTATION

Our focus is on the posterior distribution of $p(\theta|y)$ that summarize what we know about θ, such as (posterior) means, medians, modes, etc (and respective standard deviations). Knowing this allows Bayesian inference expressed as $E[g(\theta)|y]$, where $g(\theta)$ is a function of interest:

$$E[g(\theta)|y] = \int g(\theta)p(\theta|y)d\theta$$

Bar a few exceptions, it is often impossible to to evaluate the integral analytically $\Rightarrow$ simulation methods (Monte Carlo), drawing from the posterior density $p(\theta|y)$. As the number of draws (N) increases, then we can invoke the Law of Large Numbers and the Central Limit Theorem

Dynare proceeds through the following steps to arrive at the estimated posterior distribution:

1. Solves the model for a particular parameter vector θ. Currently this is a first-order (linear) solution
2. Evaluates the likelihood density $p(y \mid \theta)$ using the linear Kalman filter and assuming Gaussian shocks
3. Maximizes $p(y \mid \theta)p(\theta)$ numerically to arrive at the mode of θ (repeating 1 and 2 each time)
4. Computes an estimate of covariance matrix of the parameters, $\hat{\Sigma}_\theta$ using the result

$$\hat{\Sigma}_\theta = \left(-\frac{\partial^2 \log(p(y \mid \theta)}{\partial\theta\partial\theta'}\right)^{-1} \tag{6.1}$$

evaluated at the mode (see De Jong and Dave, chapter 15, page 403). The term in the brackets is the Hessian.

5. Output is reported at this stage (the prior mean, the estimated mode, its standard deviation and a t-test). The user can stop here.

6. Proceeds to the computation of the posterior distribution using MCMC

6.2.1 Markov Chain Monte Carlo (MCMC)

The MCMC methods sample from θ wandering over the posterior, taking most draws from high probability areas. The "Markov Chain" bit is as follows: a given draw $\theta^{(s)}$ usually depends on $\theta^{(s-1)}$. It is not easy to draw directly from $p(\theta|y)$ - we need methods that work well for any case $\Rightarrow$ MH MCMC, drawing from a candidate ("transition") distribution. From a starting value, θ_0, and thus $p(\theta_0)p(y|\theta_0)$ can be evaluated to generate draws from the posterior distribution. The idea here is to specify a transition density for a MC such that, starting from some initial value and iterating a number of times, we produce a limiting distribution which is the target distribution from which we need to sample. The intuition here is that we want to sample from the region with the highest posterior probability, but we also want to visit the whole parameter space as much as possible. Given that there is a discrepancy between the candidate and target densities, the MCMC will not take the correct draws. The *Metropolis-Hasting (MH)* algorithm corrects this by calculating an acceptance probability and eventually discarding some draws.

Because it is difficult to find a good candidate density, we usually employ a *Random Walk* Chain MH algorithm:

$$\theta^* = \theta^{(s-1)} + z$$

The sampler wanders in random directions, thus visiting most of the parameter space. z has a distribution $\alpha(\theta^*|\theta_{s-1})$ that is usually multivariate Normal and here the key choice is its covariance matrix. For each draw i, $\widehat{\theta_i} = \theta_{i-1}$ with probability $1 - r$ and $\widehat{\theta_i} = \theta_i^*$ with probability r. The acceptance probability of each new draw is then defined by:

$$r = min\left[\frac{p(\theta_i^*)p(y|\theta_i^*)}{p(\theta_{i-1})p(y|\theta_{i-1})}, 1\right] \tag{6.2}$$

The **acceptance rate** is dependent on the choice of α. One chooses α to obtain 'reasonable' acceptance rate (by adjusting σ_v^2).

- where $\alpha(\theta_i^*|\theta_{i-1}) \sim N(\theta_{i-1}, \sigma_v^2)$ and $\sigma_v^2 = c * \sigma^2 \Rightarrow$ choose the **scaling parameter** c

- Ideally 20-40% $\Rightarrow$ each move goes a reasonable distance in parameter space, but not so low as to reject too frequently

6.2.2 Testing for Convergence of MCMC

Testing for convergence of the posterior distribution is notoriously difficult, and Dynare utilizes some indicative statistics, summarized by diagrams, as recommended by Brooks and Gelman (1998). These diagrams are made up of

- 3 multivariate figures, representing convergence indicators for all parameters considered together

- 3 figures for each parameter, representing univariate convergence indicators

The diagnostics in figure 6.1 are generated by the estimation command if *mh_replic* is larger than 2000, *mh_nblocks*=2 and if option *nodiagnostic* is not used.

As an example, consider the diagnostics for NK model estimated later. The multivariate diagnostics shown below indicate that the chains converge to similar means and distributions - *Interval* refers to the interval measure, and *m2*, *m3* refer to second and third order multivariate moment measures. However, if one examines the individual parameters, the picture is somewhat mixed. Only for a subset of the parameters is there clear evidence of convergence in mean and distribution. As a minimum requirement, the multivariate diagnostics should be seen to converge to the same values. To ensure convergence of individual parameters is not straightforward as there are no analytic results that ensure convergence; the only analytic result is that if convergence is achieved under MCMC then it is to the correct posterior distribution. Recommended actions to achieve convergence are to increase the number of draws or else to increase the value of the *scale* parameter in the *estimation* command. The latter ensures that more of the parameter region is searched more regularly, but at the expense of reducing the acceptance ratio. The bottom line is that one often one has to accept that some of the convergence indicators will not always be satisfied.

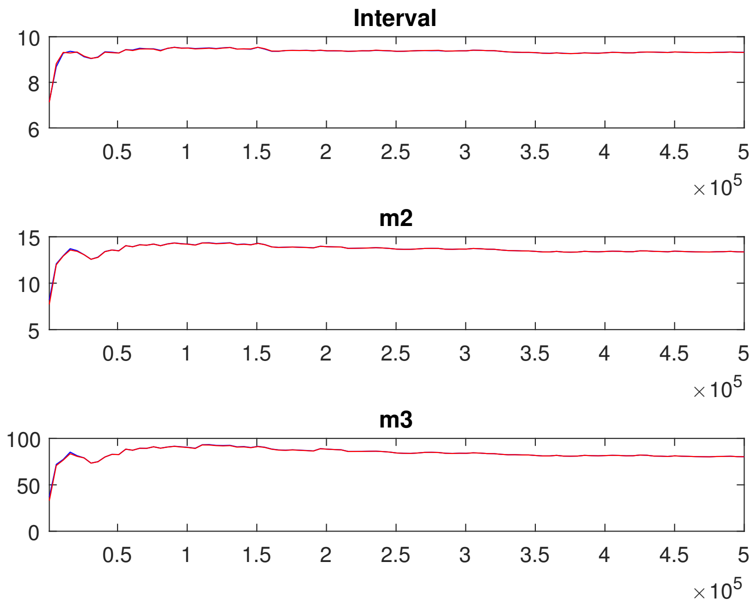


Figure 6.1: NK model estimation for US, 100,000 draws

6.2.3 Summary

Finally, below is a summary of estimation/evaluation algorithm:

1. Compute the steady state (when dealing with the non-linear solutions of the DSGE model, i.e. FOCs, other equilibrium conditions)

2. Construct a linear representation of the model and transform it into a state space framework

3. Specify prior distribution (for Bayesian method): typically the mean of prior is centred around calibrated value. Std. error reflect subjective or objective (to cover the range of existing estimated)

4. Transform the actual data to fit properties of the model (e.g. first-differencing or detrending)

5. Compute likelihood numerically via Kalman filter

6. Compute marginal likelihood for competitors. Bayes factors computed via laplace approximation in large scale system

7. Draw posterior sequences using MH: compute posterior kernel, choose a transition and use a rejection rule.

8. Check for Convergence of MCMC.

9. Construct statistics of interest from the draws (after burned in period): point estimates, credible sets, impulse responses, variance decomposition, forecast, etc.

10. Examine sensitivity of the results to choice of prior.

6.3 IDENTIFICATION ISSUES AND DIAGNOSTICS IN DSGE MODELS

One should address the question of parameter identifiability in DSGE models even before taking them to the data, as identification is a prerequisite for their usefulness. Identification failures could come from 'marginalisation' (from the model structure: i.e. mapping the deep parameters to the reduced form coefficients of the solution and mapping the solution to the population objective function) or lack of information (from the data: i.e. mapping the population to the sample objective function) and these notes provide an overview that discusses about the former. See Rothenberg (1971), Sargent (1987), Pesaran (1989) and more recently Beyer and Farmer (2004), Iskrev (2008) and Iskrev (2010), Canova and Sala (2009), and Komunjer and Ng (2011).

Detecting and tackling identification issues is difficult because it is often the case in DSGE models that the mapping from structural parameters into the model reduced form solution is highly nonlinear and the problem is compounded in large-scale models. The current approaches in the literature can mainly be summarised by three branches: 1. classical rank condition approaches that check rank deficiency of gradient matrices based on model-implied moments (e.g. Iskrev (2010)); 2. objective function approaches focusing on population moments of the data (e.g. Canova and Sala (2009) and Iskrev (2008)) and 3. observational equivalence approaches focusing on (non-linear) system matrices in the solution model (e.g. Komunjer and Ng (2011)).

The standard remedy suggested by most empirical DSGE literature is to fix some (potentially non-identifiable) parameters and re-maximise with the parameters that are well-identified. However, this can be problematic if parameters are not fixed at a consistent estimate. Canova and Sala (2009) argue that flat objective functions lead to serious biases in estimates and that fixing some of the troublesome parameters at arbitrary values may induce distortions in the distribution of parameter estimates.

However, even adding a weakly informative prior can increase the curvature of the likelihood surface. Potential under-identification can remain hidden due to the improper use of priors. The posterior distribution can be well defined as long as the joint prior distribution is proper and the only symptom of poor identification is a posterior distribution looking like the prior. As a result, it can often be unclear to what extent the reported estimates reflect information in the data instead of subjective beliefs indicated by the prior distribution. Another possible solution is to work with higher order approximations instead of linearized models, see McManus (1992).

Iskrev (2010) performs formal identification checks on the reduced form parameters and structure or deep parameters, providing a computational toolbox for local and global identification tests, as well as for testing the strength of identification that is useful for detecting parameters that are weakly identified. The procedures put forward in Iskrev (2010) and Ratto and Iskrev (2011) are the focus of these notes.

DSGE models consist of a system of non-linear equations involving a vector $\mathbf{z_t}$ of endogenous variables, a $\mathbf{u_t}$ vector of random structural shocks and a $\theta \subset \Theta$ k-dimensional vector of deep parameters. As we have seen, most applications use a linear approximation of the original model, i.e. expressing the model in terms of stationary variables linearized around their steady-states. The unique solution (if it exists), can be expressed as

$$\mathsf{z}_t = \mathbf{A}(\theta)\mathsf{z}_{t-1} + \mathbf{B}(\theta)\mathsf{u}_t \tag{6.3}$$

where $\mathbf{A}$ and $\mathbf{B}$ are functions of θ. For later use, and using Iskrev (2010) notation, denote τ as the vector collecting all the reduced-form coefficients from the DGSE model, i.e. the elements in $\mathbf{A}$, $\Omega = \mathbf{BB}'$ and the steady-state of z_t that depend on θ, respectively $\tau = [\tau_z', \tau_A', \tau_\Omega']$.

A condition for identification is that distinct values of θ imply distinct values of the probability density function of the data, as the latter contains all available sample information about the value of the parameter vector of interest θ. Usually, the distribution of the data $\mathbf{X}$ is unknown or assumed to be Gaussian and estimation of θ is based on the first two moments of the data. If $\mathbf{X}$ is not normally distributed, higher-order moments may provide additional information about θ, not contained in the first two moments. Define $\mathbf{m_T} := [\mu', \sigma_T']$ as the vector collecting the first and second order moments of the observable variables. Identification based on the mean and the variance of $\mathbf{X}$ is only sufficient but not necessary for identification with the complete distribution, so that the mapping from the population moments of the sample, $\mathbf{m_T}$, to θ is unique.

Global identification cannot, in general, be established for unique solutions of systems of non-linear equations, but local identification can be verified by means of a rank condition of the Jacobian matrix $J_T = \frac{\delta m_T}{\delta \theta'}$. Indeed, θ_0 is said to be locally identified if J_T has full column rank when evaluated at θ_0 (although this does not guarantee that the model is locally identified everywhere in the parameter space). However, studying the rank of J_T is helpful, as local identifiability ensures consistent estimation of the parameters of interest and, moreover, it will help to pinpoint the parameters that cause identification problems. For example, the column of J_T corresponding to an unidentified parameter θ_j will be a vector of zeros and the rank condition will fail. Another possibility may occur when the columns of J_T are not linearly independent, due to parameters entering the solution in a way that makes them indistinguishable (partial identification).

While numerical differentiation could be used to obtain J_T, this is unreliable when the model is highly non-linear. Instead, Iskrev (2010) proposes using analytical derivatives, employing implicit derivation, breaking down the mapping from θ to $\mathbf{m_T}$ in two steps: i) a transformation of θ to τ, ii) a transformation from τ to $\mathbf{m_T}$. The Jacobian can then be expressed as

$$J_T = \frac{\delta m_T}{\delta \tau'} \frac{\delta \tau'}{\delta \theta'} \tag{6.4}$$

The first term $J_1 = \frac{\delta m_T}{\delta \tau'}$ may be obtained by direct differentiation. Regarding the second term, Iskrev (2010) establishes a necessary condition for identification: the point θ_0 is locally identifiable only if the rank of $H_T = \frac{\delta \tau'}{\delta \theta'}$ evaluated at θ_0 is equal to k. The second term can be split into the three components of τ (Ratto and Iskrev 2011 provide details on the computation and implementation of these analytical derivatives in Dynare).[2] Note that H_T does not depend on the data, thus implying that it is possible to detect lack of identification, inherent to the structure of the DSGE model, *before* taking the model to the data. On the other hand, the rank of J_T will provide information on the identification θ given the set of observable variables and the sample size.

The above suggests a a procedure based on Monte Carlo exploration of the parameter space Θ of model parameters. The local analysis is performed in turn using the Monte Carlo realisations. This provides a 'global' prior exploration of point identification properties of DSGE models. One starts by constructing a sample drawn from Θ (discarding values that do not ensure sta-

2 This includes the case of non-linear DSGE models, for which the derivative of τ_z may not be obtained by direct differentiation, unlike linear models.

bility and determinacy). This step can be guided by use of priors, specifying a theoretically admissible range and/or a particular distribution for θ. The identifiability of each draw for θ_j is then established by studying the rank of J_T and H_T, resorting to the necessary and sufficient conditions enumerated in Iskrev (2010):

- if H_T is rank-deficient at θ_j, this particular point is unidentifiable (full rank of H_T is necessary for identification)

- if H_T has full rank but J_T does not, then θ_j cannot be identified for the particular set of observables and contemporaneous and lagged moments under consideration, i.e. given $\mathbf{X}$ and T

- weak identification issues when the columns of J_T and H_T are nearly linearly dependent (multicollinearity analysis of the re-scaled Jacobians)

Thus, a given parameter θ_j may be poorly identified because it has little impact on the reduced-from coefficients of the model ($\frac{\delta\tau}{\delta\theta_j} \approx 0$ or because its impact on the reduced-form coefficients is approximately a linear combination of $\frac{\delta\tau}{\delta\theta_i}$ of other parameters.

Performing identification analysis in Dynare along the lines of Iskrev (2010) and Ratto and Iskrev (2011) can be carried out in 'standard' or 'advanced' modes (providing a more detailed identification analysis, which however is beyond the scope of this discussion). The 'point estimate' mode is the default option in Dynare, in which local identification checks are done for the whole set, or only a subset, of the parameters in the model at a chosen central tendency measure, either at the defined prior or the estimated posterior, or at the calibrated value if no prior is declared. If priors have been defined, a Monte Carlo exploration is also available, in which the identification checks are based on samples from the prior distribution. The routines are triggered by simply using the command *identification*, which runs a check on the rank of J_T and H_T based on the priors and a list of observables defined by the user. Note that this can be carried out even before model estimation. A list of $(< options >=< values >)$ can be used to control Monte Carlo replications, see Ratto and Iskrev (2011) for more details. The output of this procedure then reveals whether or not there are identification problems, stemming from J_T and/or H_T, as illustrated below for the benchmark NK model we estimate in these Course Notes:

```
==== Identification analysis ====
Testing prior mean
All parameters are identified in the model (rank of H).
All parameters are identified by J moments (rank of J)

Monte Carlo Testing
Testing MC sample
All parameters are identified in the model (rank of H).
All parameters are identified by J moments (rank of J)
```

The upper block shows the local point identification check at the prior mean and the second is a Monte Carlo testing using 250 draws from the prior space (*identification(prior_mc=250)*). If the rank condition fails in any of these two procedures, the procedure indicates which parameters are responsible for identification problems. In particular, it reports the associated parameter(s) that are responsible for failing the column rank condition in the J_T and/or H_T and the parameter(s) with pair-wise and multi-correlation coefficients for each column of the Jacobian matrix equal to unity.

A further output of the *identification* routine is the analysis of identification 'strength', i.e., focusing on weak identification, summarized in two additional plots. The procedures are based on either the asymptotic or a moment information matrix. The first can be obtained given a sample of size T, whereas the second can be computed based on Monte Carlo simulations for samples of size T, from which sample moments of the observed variables are computed, forming a sample of N replicas of simulated moments. The corresponding information matrix is then obtained as $I_T(\theta|\mathbf{m_T}) = H_T \Sigma_{\mathbf{m_T}} H_T$, where $\Sigma_{\mathbf{m_T}}$ is the covariance matrix of simulated moments.

The 'strength' of identification for θ_i is computed as

$$s_i = \sqrt{\theta_i^2 / I_T(\theta)^{-1}_{(i,i)}} \tag{6.5}$$

which works like a t-test for θ_i. This can be decomposed into a 'sensitivity' and 'correlation' parts, the first referring to the case when weak identification arises when the moments do not change with θ_i and the second when collinearity dampens the effect of θ_i. The former is defined as

$$\Delta_i = \sqrt{\theta_i^2 \cdot I_T(\theta)_{(i,i)}} \tag{6.6}$$

which can also be normalised relative to the prior standard deviation for θ_i: $\sigma(\theta_i)$, weighting the information matrix using the prior uncertainty:

$$\Delta_i^{prior} = \sigma(\theta_i) \cdot \sqrt{I_T(\theta)_{(i,i)}} \tag{6.7}$$

This is particularly useful to distinguish the cases where (8) and (9) are singular because $\theta_i \approx 0$. It is possible to show the standard error of a parameter:

$$s.e.(\theta_i) = \frac{1}{\Delta_i} \frac{1}{\sqrt{1-\varrho_i^2}} \tag{6.8}$$

where ϱ_i denotes collinearity between the effects of different parameters so that lack of identification and a flat likelihood may be due to either $\Delta_i = 0$ or $\varrho_i = 1$.

Thus, the *identification* command plots the measures described above (also normalised relative to the prior standard deviation for θ_i), where large bars in absolute value (in the log-scale plot) imply strong identification, for the respective θ_i. Figure 6.2 shows the identification strength and sensitivity component in the moments for the NK model respectively.

In the upper panel the bars depict the identification strength of the parameters based in the Fisher information matrix normalized by either the parameter at the prior mean (blue bars) or by the standard deviation at the prior mean (yellow bars).[3] Intuitively, the bars represent the normalized curvature of the log likelihood function at the prior mean in the direction of the parameter. If the strength is 0 (for both bars) the parameter is not identified as the likelihood function is flat in this direction. The larger the absolute value of the bars, the stronger the identification. The lower panel decomposes further the effect shown above. Weak identification can be due to either i) other parameters having exactly the same effect on the likelihood, or ii) that the likelihood does not change at all with the respective parameter. This latter effect is the sensitivity effect and is plotted in the bottom panel (scaling with the prior mean in blue or the prior standard deviation in yellow).

Figure 6.3 shows an aggregate measure of how changes in the elements of the parameter vector θ impact on the model moments. The impact is measured locally using the Jacobian. The problem is that the derivatives are not scale invariant and thus not easily comparable, so Dynare uses three different normalization/standardization procedures with respect to each parameter.

3 The weighting with the prior standard deviation is particularly useful for cases where the prior mean is 0. In this case the weighting with the prior mean would falsely signal an identification strength of 0.

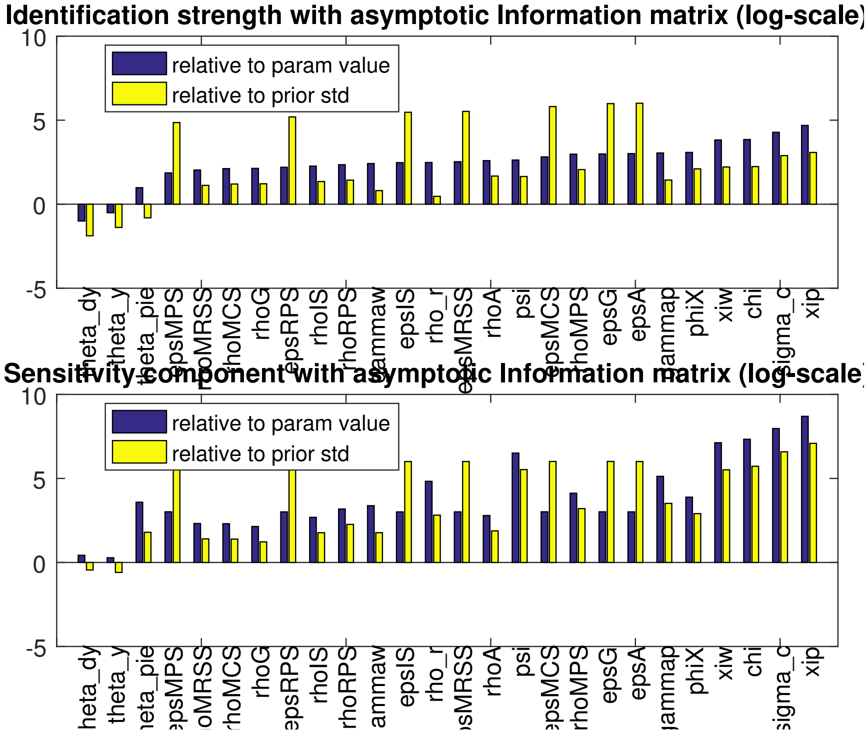


Figure 6.2: Identification Strength in NK Model- We can see that all parameters in the NK model are identified. The model parameters on the x-axis are ranked in increasing order of strength of identification. The larger the value, the stronger is the identification, as is the case for ξ_p in the rightmost diagram. In contrast, the identification for θ_{dy} in the leftmost diagram is the weakest.

The respective Jacobian refer to i) the moments matrix, indicating how well a parameter can be identified due to the strength of its impact on moments, ii) the solution matrices, indicating how well a parameter could in principle be identified if all state variables were observed, and iii) the Linear Rational Expectation (LRE) model, indicating trivial cases of non-identifiability due for example to the fact that some parameters always show up as a product in the model equations.[4]

To completely rule out a flat likelihood at the local point, one can also check collinearity between the effects of different parameters on the likelihood. If there is a exact linear dependence between a pair and among all possible combinations, their effects on the moments are not distinct and the violation of this condition must indicate a flat likelihood and lack of identification. From high

4 This does not apply to the estimation of exogenous shocks.

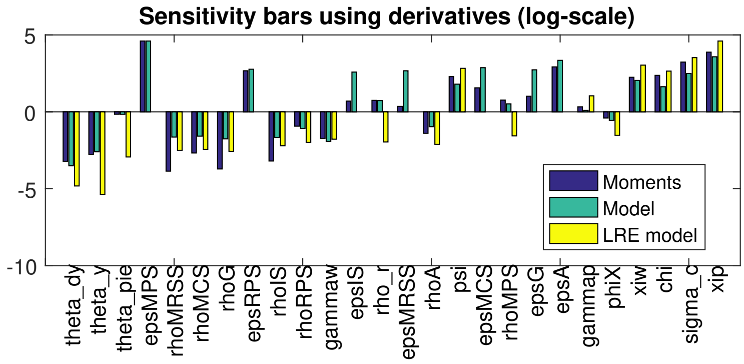


Figure 6.3: Sensitivity in NK Model- We can see that all parameters have a non-negligible effect on the moments of the NK model.

correlations to near-exact collinearity one may suspect some weak identification (due to partial identification). The details of collinearity analysis require the advanced analysis option ($advanced = 1$) which prints the results of the brute force search for the groups of parameters whose columns in the Jacobian matrix best explain each column of the Jacobian. The default maximum dimension of the group is 2 and the dimensionality is controlled by the option ($max\_dim\_cova\_group = INTEGER$). For more information on parameter similarities and high correlation Figure 6.4 plots collinearity patterns for the critical collinearities (i.e. dark yellow spots for higher correlation).

The following box prints the syntax for the default identification procedures in DYNARE IN NK: point identification at the prior mean, the MC exploration using the draws from the prior distribution and identification strength measured at the mean and weighted by the prior standard deviation.

```
varobs dyobs dcobs dinvobs dwobs robs pinfobs labobs;
identification;
identification(advanced=1,max_dim_cova_group=3);
```

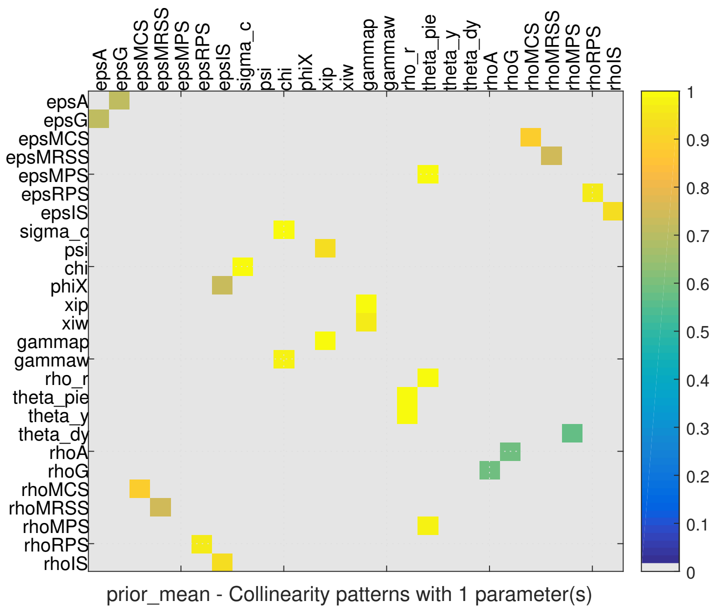


Figure 6.4: Pairwise Collinearity Patterns in NK Model - The depicted plot shows an example for a set size of one. The plot shows which linear combination of parameters shown in the columns best replicates/replaces the effect of the parameter depicted in the row on the moments of the observables. Higher values imply the relative redundancy and thus weak or un-identifiability of the parameter under consideration. The darker the brown squares are, the more critical is the collinearity between parameters. For example, the first row signifies that there is a strong correlation between the effect of technology shock standard-deviation on the NK model moments and the effect of government spending shock standard-deviation.

6.4 ESTIMATION OF THE NK AND BANKING MODELS

We estimate the models with seven observables: growth in output, consumption, investment, real wage; hours worked, inflation and the nominal interest rate. Shocks are seven exogenous AR1 shock processes to technology, government spending, the real marginal cost (the latter being interpreted as a mark-up shock), the marginal rate of substitution, an investment shock, a risk premium shock and a shock to monetary policy. So far then we do not need the shock to trend growth. But if we are to add a further financial series to represent the spread, $R_t^K - R_t$, then we do need this extra shock. Then the proposed measurement equations similar to Smets and Wouters (2007a) are:

6.4.1 Results for the NK Model

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Measurment equations use with demeaned data(prefilter=1)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
dyobs                    = log(Y/Y(-1)*(1+g));
dcobs                    = log(C/C(-1)*(1+g));
dinvobs                  = log(I/I(-1)*(1+g));
dwobs                    = log(W/W(-1)*(1+g));
labobs                   = (Hd-STEADY_STATE(Hd))/STEADY_STATE(Hd);
pinfobs                  = log(PIE);
robs                     = Rn-1;
spreadobs                = spread;
```

The data is described in the Appendix. This section sets out a strategy for the calibration and Bayesian estimation of the model parameters. First we set the nominal interest rate to 1.331847, $\Pi = 0.866298$, $g = 0.003551$, $\sigma_c = 1.5$. Together these imply a value of $\beta = 0.998$. We then set $\zeta = 7$, $\mu = 3$, $\alpha = 0.67$, $g_y = 0.18$. We calibrate θ to hit a Frisch parameter value $\delta^F = 2.5$ which is at the upper end of the empirical range (see discussion in Section 2.4). We set Calvo parameters $\xi_p = \xi_w = 0.75$ to hit observed contract frequencies. In the NK model parameters that *must* be estimated are ϕ_X, χ, γ_p, γ_w and all parameters associated with the policy rule and the AR1 shock processes.

For the banking models, as far as possible we calibrate parameters to hit target values in the balanced growth steady state set out above. The only parameter that cannot be calibrated in this way is ϵ (or κ) in (5.1).

Results for the NK model, without the spread data, obtained using mode_compute option 6 (options 4, 8 and 9 did not yield a positive definite Hessian) and 100000 MCMC draws (with convergence) took over 2 hours on my lap-top with reasonable results:

```
RESULTS FROM POSTERIOR ESTIMATION
Log data density [Laplace approximation] is 5294.385239.
```

parameters	prior mean	mode	90% HPD interval		pstdev
sigma_c	1.500	1.2158	1.0473	1.3903	0.3750
psi	2.000	2.4399	1.6353	3.2704	0.7500
chi	0.500	0.3968	0.3178	0.4753	0.1000
phiX	2.000	0.5636	0.3414	0.7820	0.7500
xip	0.500	0.6052	0.5473	0.6634	0.1000
xiw	0.500	0.5176	0.4495	0.5903	0.1000
gammap	0.500	0.3572	0.2097	0.4990	0.1000
gammaw	0.500	0.6116	0.4621	0.7606	0.1000
rho_r	0.750	0.6706	0.6090	0.7320	0.1000
theta_pie	1.500	1.9980	1.7713	2.2312	0.2500
theta_y	0.120	0.0055	-0.0153	0.0261	0.0500
theta_dy	0.120	0.1857	0.1189	0.2556	0.0500
rhoA	0.500	0.9777	0.9685	0.9870	0.2000
rhoG	0.500	0.9605	0.9506	0.9704	0.2000
rhoMCS	0.500	0.9121	0.8814	0.9431	0.2000
rhoMRSS	0.500	0.9681	0.9517	0.9849	0.2000
rhoMPS	0.500	0.2939	0.1733	0.4120	0.2000
rhoRPS	0.500	0.5987	0.4761	0.7205	0.2000
rhoIS	0.500	0.9390	0.9061	0.9727	0.2000

standard deviation of shocks

	prior mean	post. mean	90% HPD interval	
epsA	0.001	0.0093	0.0085	0.0100
epsG	0.001	0.0321	0.0295	0.0347
epsMCS	0.001	0.0147	0.0124	0.0169
epsMRSS	0.001	0.0338	0.0256	0.0423
epsMPS	0.001	0.0027	0.0024	0.0030
epsRPS	0.001	0.0052	0.0032	0.0071
epsIS	0.001	0.0166	0.0132	0.0200

Simulating the model with a second-order stochastic solution gives the variance decomposition:

```
APPROXIMATED VARIANCE DECOMPOSITION (in percent)
       epsA     epsG    epsMCS   epsMPS   epsIS   epsMRSS   epsRPS

Y      49.29     0.70     4.08     0.15    11.51    34.04     0.22
C      43.99    10.88     1.56     0.09    14.15    29.21     0.13
I      33.83     6.15     8.58     0.17    26.12    24.93     0.23
H       1.17     3.79    10.26     0.90     2.71    79.93     1.23
W      53.02     1.88    28.19     0.48    11.77     4.05     0.60
R       7.74     3.31     7.44    35.65    15.08     3.46    27.31
Q       0.76     0.24     1.19     0.99    94.54     0.48     1.79
Rn      6.16     5.23     4.53     4.04    47.00     3.57    29.47
PIE     9.32     3.37     7.81    29.90    18.48     4.68    26.44
RK      3.00     1.02     6.78    12.53    49.83     1.56    25.28
spread  0.40     0.10     2.68    16.35    40.68     0.29    39.51
CEquiv 65.85     7.96     0.87     0.00    13.72    11.59     0.00
```

So from a welfare (Ω_t) point of view the most important shocks are those to the A, MRSS, IS, and G.

6.4.2 Results for the GK Model

```
ESTIMATION RESULTS

Log data density is 5299.017734.

parameters  prior mean    mode     90% HPD interval    pstdev

sigma_c       1.500      1.3569     1.1848   1.5244    0.3750
psi           2.000      1.8580     1.0802   2.6111    0.7500
chi           0.500      0.4219     0.3521   0.4929    0.1000
```

phiX	2.000	1.5327	0.8887	2.1729	0.7500
xip	0.500	0.6573	0.6097	0.7040	0.1000
xiw	0.500	0.5076	0.4291	0.5883	0.1000
gammap	0.500	0.2769	0.1565	0.3941	0.1000
gammaw	0.500	0.6084	0.4605	0.7570	0.1000
rho_r	0.750	0.6647	0.5972	0.7325	0.1000
theta_pie	1.500	1.9528	1.7273	2.1831	0.2500
theta_y	0.120	0.0077	-0.0119	0.0269	0.0500
theta_dy	0.120	0.1804	0.1126	0.2490	0.0500
rhoA	0.500	0.9874	0.9812	0.9938	0.2000
rhoG	0.500	0.9617	0.9510	0.9723	0.2000
rhoMCS	0.500	0.9249	0.8918	0.9589	0.2000
rhoMRSS	0.500	0.9665	0.9519	0.9816	0.2000
rhoMPS	0.500	0.3081	0.1881	0.4299	0.2000
rhoRPS	0.500	0.5522	0.4315	0.6745	0.2000
rhoIS	0.500	0.8606	0.8304	0.8918	0.2000

standard deviation of shocks

	prior mean	post. mean	90% HPD interval	
epsA	0.001	0.0093	0.0085	0.0100
epsG	0.001	0.0306	0.0281	0.0331
epsMCS	0.001	0.0160	0.0133	0.0186
epsMRSS	0.001	0.0305	0.0220	0.0384
epsMPS	0.001	0.0027	0.0024	0.0030
epsRPS	0.001	0.0063	0.0038	0.0088
epsIS	0.001	0.0370	0.0294	0.0447

APPROXIMATED VARIANCE DECOMPOSITION (in percent)

	epsA	epsG	epsMCS	epsMPS	epsIS	epsMRSS	epsRPS
Y	60.29	0.46	6.58	0.27	5.66	26.59	0.16
C	60.31	5.91	3.06	0.16	7.17	23.09	0.30
I	35.63	3.62	17.21	1.67	19.02	22.36	0.49
H	1.84	4.38	16.14	1.34	2.84	72.60	0.85
W	63.27	1.03	25.87	0.43	5.87	2.96	0.57

R	5.94	2.48	9.18	38.83	25.70	3.57	14.28
Q	1.62	0.61	4.19	3.64	86.79	1.81	1.34
Rn	6.76	4.15	7.67	7.25	58.07	4.45	11.65
PIE	7.90	2.64	9.72	31.73	29.94	5.01	13.07
RK	3.08	1.32	9.34	16.10	57.50	3.08	9.58
spread	1.77	0.77	6.87	20.07	53.96	2.21	14.35
N	4.98	0.58	9.15	15.13	63.53	3.08	3.55
CEquiv	90.72	2.08	0.79	0.03	2.13	4.24	0.01
omega	1.71	1.13	7.03	18.36	65.93	1.82	4.02

6.4.3 Results for the BGG Model

ESTIMATION RESULTS

Log data density is 5283.512639.

parameters	prior mean	mode	90% HPD interval		pstdev
sigma_c	1.500	1.7425	1.4101	2.0791	0.3750
psi	2.000	0.9408	0.3546	1.5355	0.7500
chi	0.500	0.3570	0.2886	0.4267	0.1000
phiX	2.000	2.7642	1.8455	3.6465	0.7500
xip	0.500	0.7034	0.6496	0.7594	0.1000
xiw	0.500	0.4189	0.3079	0.5284	0.1000
gammap	0.500	0.3052	0.1596	0.4443	0.1000
gammaw	0.500	0.5902	0.4440	0.7409	0.1000
rho_r	0.750	0.7058	0.6570	0.7558	0.1000
theta_pie	1.500	1.9603	1.7262	2.1896	0.2500
theta_y	0.120	-0.0103	-0.0346	0.0143	0.0500
theta_dy	0.120	0.2245	0.1553	0.2929	0.0500
rhoA	0.500	0.9763	0.9643	0.9884	0.2000
rhoG	0.500	0.9666	0.9547	0.9783	0.2000
rhoMCS	0.500	0.8798	0.8302	0.9297	0.2000
rhoMRSS	0.500	0.9563	0.9355	0.9779	0.2000

rhoMPS	0.500	0.2453	0.1369	0.3517	0.2000
rhoRPS	0.500	0.8679	0.7985	0.9400	0.2000
rhoIS	0.500	0.7383	0.6457	0.8269	0.2000

standard deviation of shocks

	prior mean	post. mean	90% HPD interval	
epsA	0.001	0.0092	0.0084	0.0099
epsG	0.001	0.0288	0.0264	0.0312
epsMCS	0.001	0.0202	0.0153	0.0249
epsMRSS	0.001	0.0223	0.0158	0.0289
epsMPS	0.001	0.0026	0.0023	0.0029
epsRPS	0.001	0.0024	0.0013	0.0035
epsIS	0.001	0.0555	0.0380	0.0725

APPROXIMATED VARIANCE DECOMPOSITION (in percent)

	epsA	epsG	epsMCS	epsMPS	epsIS	epsMRSS	epsRPS
Y	8.08	1.23	80.71	0.36	1.73	7.87	0.01
C	16.01	4.34	60.53	0.45	3.94	14.64	0.09
I	3.29	0.37	89.80	1.75	0.79	3.79	0.21
H	0.56	4.86	71.36	1.08	0.13	21.98	0.03
W	5.52	0.04	92.71	0.18	1.29	0.22	0.04
R	1.50	1.34	7.21	72.61	0.75	2.44	14.14
Q	0.17	0.01	1.79	1.55	96.21	0.17	0.11
Rn	4.87	5.24	49.21	6.13	3.67	17.47	13.41
PIE	4.04	4.22	37.98	24.11	2.87	14.07	12.71
RK	1.27	0.17	18.68	34.66	43.53	1.07	0.61
spread	0.44	0.53	18.36	44.94	34.93	0.29	0.51
CEquiv	9.39	1.83	82.24	0.04	4.43	2.08	0.00

6.5 BAYESIAN MODEL COMPARISONS

First, let us go back to Bayes Rule

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} \propto p(y|\theta)p(\theta)$$

So far we have not needed the unconditional density $p(y)$ to maximize $p(\theta|y)$ wrt θ. This is computed by integrating over the prior distribution to obtain

$$p(y) = \int_{\Theta} p(y|\theta)p(\theta)d\theta$$

For a particular model i from a number of alternatives, say m_i, we can define a density conditional on this model

$$p(y|m_i) = \int_{\Theta} p(y|\theta, m_i)p(\theta, m_i)d\theta$$

where $p(\theta, m_i)$ is the prior for that model. We refer to $p(y|m_i)$ as the *marginal likelihood* associated with model m_i.

Bayesian inference also allows a framework for comparing alternative and potentially misspecified models based on their marginal likelihood. For a given model $m_i \in M$ and common dataset, the latter is obtained by integrating out vector θ,

$$p\left(y|m_i\right) = \int_{\Theta} p\left(y|\theta, m_i\right) p\left(\theta|m_i\right) d\theta$$

where $p_i\left(\theta|m_i\right)$ is the prior density for model m_i, and $p\left(y|m_i\right)$ is the data density for model m_i given parameter vector θ. To compare models (say, m_i and m_j) we calculate the posterior odds ratio which is the ratio of their posterior model probabilities (or Bayes Factor when the prior odds ratio, $\frac{p(m_i)}{p(m_j)}$, is set to unity):

$$PO_{i,j} = \frac{p(m_i|y)}{p(m_j|y)} = \frac{p(y|m_i)p(m_i)}{p(y|m_j)p(m_j)} \quad (6.9)$$

$$BF_{i,j} = \frac{p(y|m_i)}{p(y|m_j)} = \frac{\exp(LL(y|m_i))}{\exp(LL(y|m_j))} \quad (6.10)$$

defining the log-likelihood

$$LL(y|m_i) \equiv log(p(y|m_i))$$

noting that $x = \exp(\log x)$). Components (6.9) and (6.10) provide a framework for comparing alternative and potentially misspecified models based on their marginal likelihood. Such comparisons are important in the assessment of rival models.

Given Bayes factors we can easily compute the model probabilities $p_1, p_2, \cdot\cdot\cdot p_n$ for n models. Since $\sum_{i=1}^{n} p_i = 1$ we have that

$$\frac{1}{p_1} = \sum_{i=2}^{n} BF_{i,1}$$

from which p_1 is obtained. Then $p_i = p_1 BF(i,1)$ gives the remaining model probabilities. The MATLAB programme, **modelcomparison.m**, computes these probabilities give the the log-likelihood values from the competing models. The following table provides a formal Bayesian comparison of the benchmark NK model with both the GK and BGG models, using both the second stage log-likelihood (preferable, as it pertains to the recovered posterior density).

	NK	GK	BGG
LLs (2^{nd} stage)	5294.385239	5299.017734	5283.512639
prob.	0.0096	0.9904	0.0000

Table 6.1: Marginal Log-likelihood Values and Posterior Model Odds

Our model comparison analysis suggests that the presence of BGG financial frictions not does improve model fit, but NK frictions do. Recall, however, that the results are sensitive to the length of sample and choice of observables (also see exercises in Slides).

A limitation of the likelihood race methodology is that the assessment of model fit is only *relative* to its other rivals with different restrictions. The outperforming model in the space of competing models may still be poor (potentially misspecified) in capturing the important dynamics in the data. To further evaluate the *absolute* performance of one particular model against data, in a later section we compare the model's implied characteristics with those of the actual data (a comparison with benchmark DSGE-VAR is also useful).

6.6 SECOND MOMENT COMPARISONS WITH DATA

The focus on various alternative specifications seeks to address some of the concerns with Bayesian model comparisons pointed out by Sims (2003). By estimating a large number of model variants, this method intends to complete the space of competing models and to compute posterior odds that take into consideration other (seemingly irrelevant) aspects of the specification. One obvious pitfall or limitation of this methodology is that the assessment of how fit a model is only relative to its other rivals with different restrictions. The outperforming model in the space of competing models may still be poor in capturing the important dynamics in the data.

Sims (2003) argues when weighing the evidence in favor of a particular characteristic of the model failing to account for other aspects of the specification can lead to disparate inference. In other words, the Bayesian model comparison (based on Bayes factors and posterior odds) is criticized on the basis of the argument that the models considered are too sparse. In such cases, posterior odds may lead to extreme outcomes and may also be highly dependent on the prior distribution. Sims (2003) proposes that the possible solution is to 'fill the space of models' to make the model comparison more robust. Recent work by Del Negro *et al.* (2007) also seeks to address the lack of continuity in the model space when comparing DSGE models relative to BVARs, by combining data with artificial observations generated by the model. To further evaluate the absolute performance of one particular model (or information assumption) against data, in a later section we compare the model's implied characteristics with those of the actual data and with a benchmark DSGE-VAR model.

The summary statistics such as first and second moments have been standard for researchers to use to validate models in the literature on DSGE models, especially in the RBC tradition. As the Bayes factors (or posterior model odds) are used to assess the relative fit amongst a number of competing models, the question of comparing the moments is: can the models correctly predict population moments, such as the variables' volatility or their correlation, i.e. to assess the absolute fit of a model to macroeconomic data.

Following Schorfheide (2000), let y^{rep} be a sample of observation that one could have observed in the past or that one might observe in the future. One can derive the sampling distribution of y^{rep} given the current state of knowledge using the Bayes theorem: $p(y^{rep}|y) = \int L(y^{rep}|\theta)p(\theta|y)d\theta$. Assume that $T(y)$ is a test quantity that reflects an aspect of the data (moment) that one wants to check, e.g. correlation between output and inflation or the output volatility. In order to assess whether the estimated model can replicate population moments, one sequentially generates draws from the posterior distribution, $p(\theta|y)$ and the predictive distribution $p(y^{rep}|y)$ so that the predictive $T(y^{rep})$ can be computed.

For the simulation and computation of moments, Dynare assumes that the shocks follow a normal distribution. In a stochastic set-up, shocks are only allowed to be temporary. A permanent shock cannot be accommodated because of the need to stationarize the model. Also the expectations of future shocks in a stochastic model must be zero. But in Dynare we can make the effect of the shock propagate slowly throughout the economy by specifying the shock's process and introducing a "latent shock variable", that affects the model's true exogenous variable, which is itself an AR(1). In a stochastic framework, these exogenous variables take random values in each period. In Dynare, these random values follow a normal distribution with zero mean, but we can (and have to) specify the variability of these shocks (within the *shock;...end;* block). So setting *period=1000* when simulating the model specifies that the model is simulated over 1000 periods, where Dynare computes the path of variables over a 1000 period horizon by solving all the equations for every period, and this can be used to compute the (empirical) moments of the simulated variables (i.e. simulated model solutions).

To obtain the model-generated moments based on the real world data (i.e. posterior distribution), simply use the *stoch_simul* keyword after the estimation command. Table 6.2 presents some selected second moments implied by the above estimations of models 1 and 2. In particular, we compute these model-implied statistics by solving the models at the posterior means obtained from estimation. The results of the models' second moments are compared with the second moments in the actual data to evaluate the models' empirical performance.

We have so far considered autocorrelation only up to order 1. To further illustrate how the estimated models capture the data statistics and persistence in particular, we now plot the autocorrelations up to order 10 of the actual data and those of the endogenous variables generated by the model variants. Using the *stoch_simul* keyword and argument, e.g., *ar=10* we produce autocorrelations up to order 10. All simulation outputs from Dynare are stored in *FILENAME_results.mat* in the working directory. So we reload it to extract useful fields (stored in the struc. array *oo_*) For instance, the simulated autocorrelation function can be found on the diagonal of the field *oo_.autocorr*. To compute and plot the sample ACFs from the data, we need subfunctions

Model	Output	Investments	Wage	Int. Rate	Inflation	Spread
			Means			
Data	0.3551	0.3635	0.3413	1.3318	0.8663	0.8031
NK	0.3544	0.3544	0.3544	1.2882	0.8277	-0.0027
GK	0.3544	0.3544	0.3544	1.3022	0.8402	0.6938
BGG	0.3544	0.3544	0.3544	1.2866	0.8202	0.7105
			Standard Deviation			
Data	0.8017	2.1127	0.7531	0.9493	0.5855	0.5435
NK	0.8393	2.2463	1.9049	1.4562	0.8280	1.1764
GK	0.7953	2.7529	2.7893	0.7787	0.6756	1.3578
BGG	0.9292	3.1188	1.2070	0.8669	0.8432	1.4948
			Cross-correlation with Output			
Data	1.00	0.6791	0.0167	-0.0675	-0.1772	0.1211
NK	1.00	0.7621	0.5410	-0.0216	-0.0584	0.1514
GK	1.00	0.7538	0.4425	-0.0840	-0.0565	0.3253
BGG	1.00	0.7728	0.5190	0.1375	0.2324	0.5295

Table 6.2: Selected Moments of the Model Variants

acfcomp.m and **autocov.m**. Finally, in the working directory, **acfs_plot.m** plots the sample ACFs and estimated ACFs.

6.7 IMPULSE RESPONSE FUNCTIONS

The following figure 6.5 depicts the mean responses corresponding to a positive one standard deviation technological shock. The endogenous variables of interest are the observables in the estimation and each response is for a 10 period (2.5 years) horizon. All impulse responses are computed simulating the vector of DSGE model parameters at the posterior mean values computed from estimation.

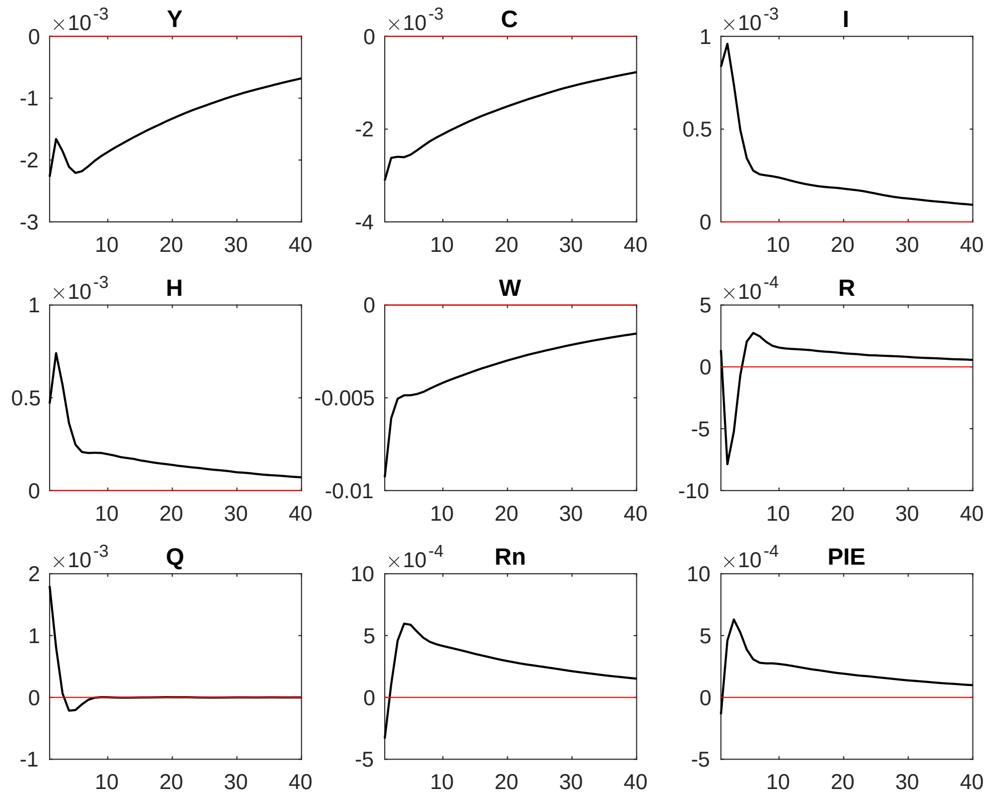


Figure 6.5: NK model IRFs- Technology Shock

6.8 VARIANCE AND HISTORICAL DECOMPOSITIONS

As explained above, impulse response functions allow us to analyse the effects of a shock to one of the endogenous variables on to the other variables. Variance decompositions, on the other hand, decomposes the variation in each endogenous variable into each shock to the system, thus providing information on the relative importance of each disturbance as a source of variation for each variable. Historical decompositions, in turn, can be used for counterfactual simulations. The data can be decomposed into the sum of a baseline forecast and the contribution of all shocks. This allow us to analyse how the data would have evolved if a shock or a combination of shocks are shut down (i.e., their contribution is zero).

The linear state-space representation of the model solution is given by

$$\begin{aligned} X_{t+1} &= AX_t + B\varepsilon_{t+1} \\ Y_t &= CX_t \end{aligned}$$

where X_t is the potentially unobservable state vector and Y_t is the vector of the observables. The historical decomposition stems from the Moving Average (MA) representation of the model state space

$$Y_h = \sum_{j=0}^{h} d_j \varepsilon_{h-j} + CA^h X_0 \tag{6.11}$$

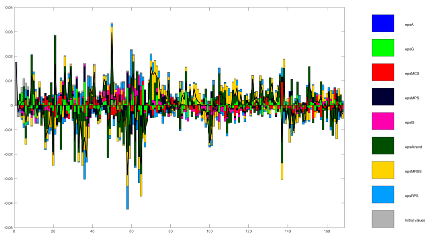
for $h = 1, ..., T$. Note that each d_j is a matrix, with its ith column $d_{i,j}$ multiplying the ith shock; if we further define the effect of all the ith shocks on Y_h as $Y_{i,h} = \sum_{j=0}^{h} d_{i,j}\varepsilon_{i,h-j}$, then we can decompose Y_h as

$$Y_h = \sum_{i=0}^{r} Y_{i,h} + CA^h X_0 \tag{6.12}$$

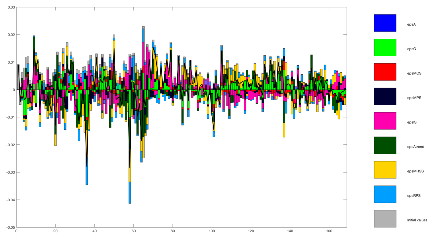
where r is the number of shocks.

During the course of the estimation, Dynare automatically produces the Kalman filter estimates of all the $\{X_t\}$, and for the final set of parameters (either the mode, or the average over the MCMC iterations) it also calculates the smoothed estimates of the $\{X_t\}$. From these it also calculates the smoothed estimates of all the shocks. These latter, together with the smoothed estimates of X_0 are then used to calculate each of the individual terms of (6.12), with the last term, $CA^h X_0$, shown on the historical decomposition graphs as the effect of 'initial values'.

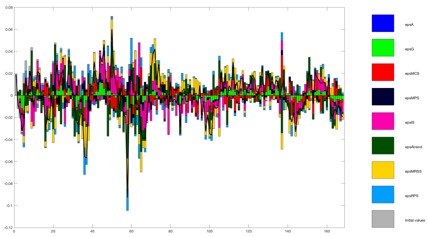
In Dynare, variance decomposition for the specific periods (specified in [ ] after the option) can be carried out by using the option *conditional_variance_decomposition*. The decomposition for a given sample according to the estimated model can be computed using the command *shock_decomposition* which must be followed by the *estimation* statement (unless one specifies to use the calibrated parameters with the option *parameter_set*). The following figure provides, for inflation, historical decomposition of our estimation samples to each of the estimated shocks (based on the estimated posterior modes obtained using the NK linear model above). Figures 6.6a– 6.6f decompose the historical time series with the NK model using the estimated parameters. In particular it presents the contribution of the respective shocks to the deviation of the smoothed inflation from its steady state with the coloured bars. Note that the figures below also plot the initial conditions which represent the distance between the rational expectation solution from its steady state before the shock arrives. In other words, these initial conditions refer to the part of the deviations from its steady state explained by the unknown initial value of the state variables. The influence of initial values usually dies out relatively quickly but its persistence depends on how much persistence the model has.



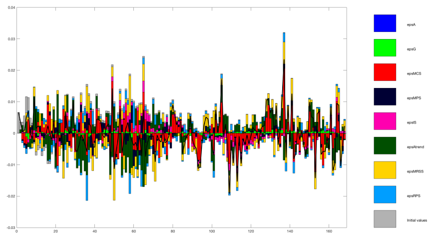
(a) Output



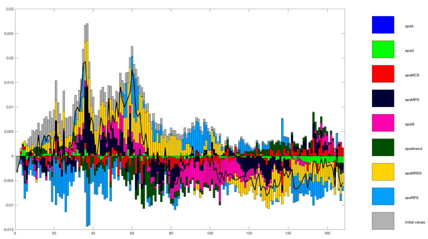
(b) Consumption



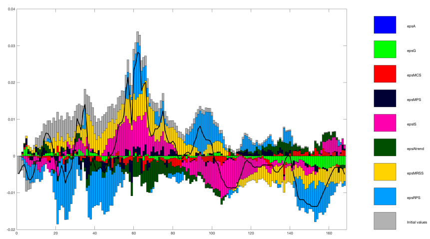
(c) Investment



(d) Inflation



(e) Interest rate



(f) Exchange rate

Figure 6.6: Historical Decompositions, US 1966-2008

7 OPTIMAL POLICY

This section first sets out the general procedure for computing optimal Taylor-type commitment rules in a DSGE models with any number of instruments such as the nominal interest rate (considered up to now), fiscal instruments (government spending, tax rates) and a macro-prudential instrument. Then we study a macro-prudential instrument in the context of the GK model with outside equity.

7.1 THE GENERAL OPTIMAL POLICY PROBLEM

This section first considers the general optimal policy problem where the policymaker has a number of instruments and sets out to maximize a general discounted welfare criterion subject to the constraints of a DSGE model. If the policymaker is able to commit, the setting of instruments can be conducted in terms of the ex ante optimal policy. If the expected discounted household utility is chosen as the welfare criterion this becomes the well-known Ramsey problem. A problem with such a solution is that it involves a complex rule even for quite simple NK models. Much of the optimal policy literature therefore focuses on simple Taylor-type commitment rules that are optimized so as to come close to mimicking the Ramsey solution and this is the approach of this course. In the absence of an ability to commit the policymaker must set policy to be time-consistent. Before proceeding with our treatment of simple rules, we first briefly review the Ramsey and time-consistent solutions.

7.1.1 The Ramsey Problem

We consider a model as a special case of the following general setup recognized by Dynare in non-linear form

$$Z_t = J(Z_{t-1}, X_t, w_t, \epsilon_t) \tag{7.1}$$

$$\mathbb{E}_t X_{t+1} = K(Z_t, X_t, w_t) \tag{7.2}$$

where $\mathsf{Z}_t, \mathsf{X}_t$ are $(n-m)\times 1$ and $m \times 1$ vectors of backward and forward-looking variables, respectively, ϵ_t is a $\ell \times 1$ i.i.d shock variable and w_t is an $r \times 1$ vector of instruments. Under perfect information all variables dated t or earlier are observed at time t including shocks.

Now define

$$\mathsf{y}_t \equiv \begin{bmatrix} \mathsf{Z}_t \\ \mathsf{X}_t \end{bmatrix}$$

Then, as in Dynare User Guide, chapter 7, (7.1) and (7.2) can be written

$$\begin{aligned} \mathbb{E}_t[f(\mathsf{y}_t, \mathsf{y}_{t+1}, \mathsf{y}_{t-1}, \mathsf{w}_t, \epsilon_{t+1})] &= 0 \qquad (7.3) \\ E_t[\epsilon_{t+1}] &= 0 \\ E_t[\epsilon_{t+1}\epsilon'_{t+1}] &= \Sigma_\epsilon \end{aligned}$$

This is quite general in that y_t can be enlarged to include lagged and forward-looking variables.

The general problem is to maximize at time 0, $\Omega_0 = E_0\left[\sum_{t=0}^{\infty}\beta^t u(\mathsf{y}_t, \mathsf{y}_{t-1}, \mathsf{w}_t)\right]$ subject to (7.3) given initial values Z_0.[1] To carry out this problem write the Lagrangian

$$L = E_0\left[\sum \beta^t [u(\mathsf{y}_t, \mathsf{y}_{t-1}, \mathsf{w}_t) + \lambda'_{t+1} f(\mathsf{y}_t, \mathsf{y}_{t+1}, \mathsf{y}_{t-1}, \mathsf{w}_t, \epsilon_{t+1}]\right] \qquad (7.4)$$

where λ_t is a column vector of multipliers associated with the n constraints defining the model. First-order conditions are given by

$$\begin{aligned} E_0\left[\frac{\partial L}{\partial \mathsf{w}_t}\right] &= E_0\left[u_3(\mathsf{y}_t, \mathsf{y}_{t-1}, \mathsf{w}_t) + \lambda'_{t+1} f_4(\mathsf{y}_t, \mathsf{y}_{t+1}, \mathsf{y}_{t-1}, \mathsf{w}_t, \epsilon_{t+1})\right] = 0 \qquad (7.5) \\ E_0\left[\frac{\partial L}{\partial \mathsf{y}_t}\right] &= E_0[u_1(\mathsf{y}_t, \mathsf{y}_{t-1}, \mathsf{w}_t) + \beta u_2(\mathsf{y}_{t+1}, \mathsf{y}_t, \mathsf{w}_{t+1}) + \lambda'_{t+1} f_1(\mathsf{y}_t, \mathsf{y}_{t+1}, \mathsf{y}_{t-1}, \mathsf{w}_t, \epsilon_{t+1}) \\ &+ \frac{1}{\beta}\lambda'_t f_2(\mathsf{y}_{t-1}, \mathsf{y}_t, \mathsf{y}_{t-2}, \mathsf{w}_{t-1}, \epsilon_t) + \beta\lambda'_{t+2} f_3(\mathsf{y}_{t+1}, \mathsf{y}_{t+2}, \mathsf{y}_t, \mathsf{w}_{t+1}, \epsilon_{t+1})] \\ &= 0 \qquad (7.6) \end{aligned}$$

where the subscripts in $\{u_i, f_i\}$ refer to the partial derivatives of the ith variable in u, f.

Now partition $\lambda_t = [\lambda_{1,t}\,\lambda_{2,t}]$ so that $\lambda_{1,t}$, the co-state vector associated with the backward-looking component of (7.5), namely (7.1), is of dimension $(n-m)\times 1$ and $\lambda_{2,t}$, the co-state vector associated with the forward-looking component of (7.5), namely (7.2), is of dimension $m \times 1$.

1 The treatment here is for a zero-growth steady state.

An important optimality condition is:

$$\lambda_{2,0} = 0; \text{ (ex ante optimal)} \tag{7.7}$$
$$\lambda_{2,0} = \lambda_2; \text{ ('timeless' solution)} \tag{7.8}$$

where λ_2 is the deterministic steady state of $\lambda_{2,t}$. To complete our solution we require $2n$ boundary conditions. Then together with (7.7) or (7.8), Z_0 given are n of these. The 'transversality condition' $\lim_{t\to\infty} \lambda_t = \lambda$ gives us the remaining n (see Currie and Levine (1993). chapter 4).

Thus in order to achieve optimality the policy-maker sets $\lambda_{2,0} = 0$ at time $t = 0$. As is well-known optimal policy sees an initial jump in the inflation rate even in the absence of any shocks. The timeless solution removes this feature and imposes a time-invariance on the solution (see Adam (2011), for example). For the ex ante optimal policy, at time $t > 0$ there then exists a gain from reneging by resetting $\lambda_{2,t} = 0$. Thus there is an incentive to renege that exists at all points along the trajectory of the optimal policy by re-optimizing in this fashion. This essentially is the *time-inconsistency problem* facing stabilization policy in a model with structural dynamics. An easier understanding of the difference between the fully optimal and the timeless solution is obtained by evaluating the linear-quadratic (LQ) approximation about the steady state of the optimum to the problem (assuming that the optimal solution converges to a steady state). The welfare approximation is given by by a discounted sum of terms, starting at time $t = 0$, each of which is a quadratic form in Δy_t and Δw_t, where the latter are deviations of y_t and w_t about their steady states, and the weighting matrix of the quadratic form is given by the second derivatives of the Lagrangian divided by 2; in addition the welfare approximation contains a linear term in Δy_0 (but not any subsequent Δy_t). This latter term is irrelevant in a system that is purely backward looking, and merely represents the initial condition of the system. However for a forward-looking system, the initial value of ΔX_0 is dependent on the instrument, and is the reason why the fully optimal policy sees an initial deterministic jump even when ΔZ_0 starts at its steady state value. The timeless solution ignores the linear term in the welfare approximation.

7.1.2 Time-Consistent (Discretionary) Policy

To evaluate the time-consistent (discretionary) policy we write the expected loss $\Omega_t(Z_t)$ at time t given observed Z_t, in Bellman form as[2]

$$\Omega_t(Z_t) = \mathbb{E}_t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}u(\mathrm{y}_t,\mathrm{y}_{t-1},\mathrm{w}_t)\right] = u(\mathrm{y}_t,\mathrm{y}_{t-1},\mathrm{w}_t) + \beta\mathbb{E}_t\left[\Omega_{t+1}(Z_{t+1})\right] \quad (7.9)$$

Define the value function

$$V(\mathrm{Z}_t) = \max_{\{w_t\}}\left\{u(\mathrm{y}_t,\mathrm{y}_{t-1},\mathrm{w}_t) + \beta\mathbb{E}_t\left[V_{t+1}(\mathrm{Z}_{t+1})\right]\right\} \quad (7.10)$$

Then Bellman's equation is

$$V_t(\mathrm{Z}_t) = \max_{\{w_t\}}\mathbb{E}_t\left\{u(\mathrm{y}_t,\mathrm{y}_{t-1},\mathrm{w}_t) + \beta\mathbb{E}_t\left[V_{t+1}(\mathrm{Z}_{t+1})\right]\right\} \quad (7.11)$$

In other words Ω_t is maximized at time t, subject to the model constraints, in the knowledge that a similar procedure will be used to minimize Ω_{t+1} at time $t+1$. The dynamic programming solution then seeks a stationary *Markov Perfect* solution of the form $\mathrm{w}_t = F(\mathrm{Z}_t)$, $\mathrm{X}_t = G(\mathrm{Z}_t)$ and $V_t(Z_t) = V(Z_t)$.[3] There is no computationally feasible general algorithms to find the time-consistent solution, other than when the system is only backward-looking (when the fully optimal and time-consistent policies are identical), except for the case when the system is linear and the loss function is quadratic.[4] One commonly used approach for the general problem is to assume that policymakers can commit to direct the system to the steady state of the full optimum, but might not be trusted with regard to stabilization around that optimum. Then the time-consistent problem reduces to finding the LQ approximation about the optimum, and then solving the Bellman problem. The same LQ approximation is also used to check whether the assumption of convergence of the optimum to a steady state is correct, by evaluating the 2nd order conditions that depend on the Riccati equation for the fully optimal case.

2 This applies only to the zero-growth steady state. To stationarize the problem for a trended balanced-growth steady state see Appendix B.

3 See Currie and Levine (1993) and Söderlind (1999) for a LQ treatment of this problem under perfect information and Levine *et al.* (2012) under imperfect information. But see Dennis and Kirsanova (2013) for the possibility of multiple equilibria.

4 A Markov-perfect time-consistent solution requires global methods which are, as yet, not feasible for a medium-sized NK model with many state variables. For a small RBC model however see Dennis and Kirsanova (2015)

7.1.3 Optimized Simple Rules

Optimal policy in the form of the Ramsey solution can be expressed as $\mathsf{w}_t = f(\mathsf{Z}_t, \lambda_{2,t})$. This poses problems for the implementability of policy in terms of complexity and the observability of elements of Z_t (such as the technology process A_t, but more importantly $\lambda_{2,t}$). The macroeconomic policy literature therefore focuses on simple rules, using the Ramsey solution as a benchmark.

The general optimal policy problem seeks an optimized simple rule in which the vector of instruments w_t respond to an observed subset of macroeconomic variables in a prescribed (for example log-linear) fashion. All our rules take the log-linear form

$$\log \mathsf{w}_t = D \log \mathsf{y}_t \tag{7.12}$$

where we define $\log \mathsf{w}_t \equiv [\log w_{1,t} \log w_{2,t}, \cdots, \log w_{r,t}]'$ over r instruments, and similarly for $\log \mathsf{y}_t$, and the matrix D selects a subset of y_t from which to feedback. Again this is quite general in that y_t can be enlarged to include lagged and forward-looking variables.

The optimized simple rules then defines the inter-temporal welfare loss at time t in Bellman form (7.9) (again ignoring long-term trend growth for now), sets steady-state values for instruments w_t, denoted by w, computes a second-order solution for a particular setting of w and solves the maximization problem at $t = 0$,

$$\max_{\mathsf{w},D} \Omega_0(\mathsf{Z}_0, \mathsf{w}, D) \tag{7.13}$$

given initial values Z_0. In a purely stochastic problem we put $\mathsf{Z}_0 = \mathsf{Z}$, the steady state of Z_t, maximizing the conditional welfare at the steady state. In a purely deterministic problem there is no exogenous uncertainty and the optimization problem is driven by the need to return from Z_0 to its steady state, Z. We now examine optimal monetary policy conducted in terms of the Ramsey solution, discretion and either of two simple Taylor interest rate rules used up to now, (2.54) and (2.55) which are special cases of (7.12). To allow for the possibility that $\rho_r = 1$, we re-parameterize the feedback coefficients by setting $\alpha_\pi = (1 - \rho_r)\theta_\pi$, $\alpha_y = (1 - \rho_r)\theta_y$ and optimizing with respect to ρ_r, α_π and α_y.

To proceed we write the inter-temporal welfare loss at time t in Bellman form as

$$\Omega_t = u_t + \beta \mathbb{E}_t [\Omega_{t+1}]$$

It is now established that the Ramsey-solution to NK models such as ours sets $\Pi = 1$ in the steady state. Optimized rules then set $\Pi = 1$ and optimize a *second-order approximation of the mean of* Ω_t over ρ_r, θ_π and $\theta_{r,y}$. In what follows we focus on the purely stochastic problem (as defined above) and therefore start at the steady state.

7.2 JR PREFERENCES

The NK model up to now with a SW utility function displays a strong *wealth effect* in response to a positive technology shock. As a result household reduce their hours relative to the steady state and "consume" more leisure. Hours and output then do not co-move, a feature we do see in the data.

An alternative functional form for utility found in the literature from Jaimovich and Rebello (2008) controls the wealth effect. It takes the form:

$$U_t = \frac{(C_t - \kappa H_t^{\theta} \Xi_t)^{1-\sigma_c} - 1}{1-\sigma_c} \tag{7.14}$$

$$\Xi_t = C_t^{\gamma} \Xi_{t-1}^{1-\gamma}; \quad \gamma \in [0,1] \tag{7.15}$$

The parameter κ can be set to target $\bar{H}$ (as we did using ϱ with the Cobb-Douglas function previously). The parameter θ can be set to target the elasticity of labour supply with respect to the real wage.[5] This leaves γ to control for wealth effects. With $\gamma = 1$ we have a utility function of the form

$$U_t = \frac{(C_t(1 - \kappa H_t^{\theta}))^{1-\sigma_c} - 1}{1-\sigma_c} \tag{7.16}$$

A CD utility function is less flexible in that it can only target one steady state outcome $H = \bar{H}$ whereas the JR utility function can target labour supply elasticity and (as we shall see) wealth effects.

From Appendix 1.2, the first-order conditions for the household now become:

$$\text{Euler Consumption} : 1 = R_t \mathbb{E}_t \left[\Lambda_{t,t+1}\right] \tag{7.17}$$

$$\text{Stochastic Discount Factor} : \Lambda_{t,t+1} \equiv \beta \frac{\lambda_{t+1}}{\lambda_t} \tag{7.18}$$

$$\text{where} : \lambda_t = U_{C,t} - \gamma \mu_t \frac{\Xi_t}{C_t} \tag{7.19}$$

$$\text{and} : \mu_t = -U_{\Xi,t} + \beta(1-\gamma)\mathbb{E}_t \frac{\mu_{t+1}\Xi_{t+1}}{\Xi_t} \tag{7.20}$$

5 See Bilbiie (2009) and Bilbiie (2011) for details.

$$\text{Labour Supply} : \frac{U_{H,t}}{\lambda_t} = -W_t \tag{7.21}$$

$$\begin{aligned}\text{Investment FOC} : & \, Q_t(1 - S(X_t) - X_t S'(X_t)) \\ & + \mathbb{E}_t \left[\Lambda_{t,t+1} Q_{t+1} S'(X_{t+1}) X_{t+1}^2 \right] = 1\end{aligned} \tag{7.22}$$

$$\text{Capital Supply} : \mathbb{E}_t \left[\Lambda_{t,t+1} R_{t+1}^K \right] = 1 \tag{7.23}$$

where R_t^K is the gross return on capital given by

$$R_t^K = \frac{[r_t^K + (1-\delta)Q_t]}{Q_{t-1}} \tag{7.24}$$

7.3 MACRO-PRUDENTIAL REGULATION

The GK model with outside equity can be used to examine the effects of financial macro-prudential regulation alongside conventional monetary policy. We consider a rule that directly regulates capital requirements in the form of leverage ($lever_t$), defined as the proportion of total loans to inside plus outside equity (net worth) defined as:

$$lever_t = \frac{Q_t S_t}{N_t + q_t E_t} \tag{7.25}$$

Then the rule take one of two forms

$$\begin{aligned}\log\left(\frac{lever_t}{lever}\right) &= \rho_{lever} \log\left(\frac{lever_{t-1}}{lever}\right) - lever_y \log\left(\frac{Y_t}{Y}\right) \\ &+ lever_{spread} \log\left(\frac{1 + \text{spread}_t}{1 + \text{spread}}\right)\end{aligned} \tag{7.26}$$

$$\begin{aligned}\log\left(\frac{lever_t}{lever}\right) &= \rho_{lever} \log\left(\frac{lever_{t-1}}{lever}\right) - lever_y \log\left(\frac{Y_t/Y}{Y_t^F/Y^F}\right) \\ &+ lever_{spread} \log\left(\frac{1 + \text{spread}_t}{1 + \text{spread}}\right)\end{aligned} \tag{7.27}$$

with $lever_y, lever_{spread} > 0$ so that leverage is require to respond counter-cyclically (pro-cyclically) to output (spread) where spread $\equiv R_t^K - R_t$ as before. The bank does not optimize with respect to e_t, and the rule then replaces the bank's first-order condition for the decentralized choice of x_t, $(1+\lambda_t)\mu_{e,t} = \Theta_t'\lambda_t$.

The calibration chosen is for the high decentralized equity ratio $x = 0.15$. Table 7.1 sets out the results for the regulation rule with a conventional Taylor rule for monetary policy. Since our rule regulates equity (the capital requirement) in a pro-cyclical manner and responds positively to spread for the direct control of x we see a marked increase in the volatility of equity which for higher values of the feedback coefficients involves a significant welfare cost. Table 7.2 calculates the welfare optimized form of the rule with respect to the feedback=$lever_y = lever_{spread}$.

In *stationarized form* (See Section 2.6.1) with a shock to trend, the inter-temporal welfare is given by

$$\begin{aligned}
\Omega_t &= U_t + \mathbb{E}_t\left[(1+g_{t+1})\beta_{g,t+1}\Omega_{t+1}\right] \text{ where} \\
\beta_{g,t} &\equiv \beta(1+g_t)^{-\sigma_c} \text{ (growth-adjusted discount factor)}
\end{aligned}$$

Given a particular equilibrium for C_t and H_t and single-period utility, $U_t = U(C_t, C_{t-1}, H_t)$ we then compute $CEequiv_t$, the increase in the given by a 1% increase in consumption, by defining the variable:

$$\begin{aligned}
CEequiv_t &\equiv U_t(1.01\,C_t, 1.01\,C_{t-1}/(1+g), H_t) - U_t \\
&+ \mathbb{E}_t\left[(1+g_{t+1})\beta_{g,t+1}CEequiv_{t+1}\right]
\end{aligned}$$

Then we use the deterministic steady state of $CEequiv_t$, CE, to compare welfare outcomes: for two welfare outcomes, W_1 and W_2, we define $ce \equiv \frac{W_1 - W_2}{CE}$ reported in Table below.

The welfare cost of regulation in consumption equivalent percentage units of the deterministic steady-state (CE) is reported and is significant at around 0.13% - 0.24%consumption equivalent for our chosen calibration of parameters and shocks. The gain from using an optimized form of the regulatory rule with feedback coefficients $lever_y = lever_{spread} = 0.18 - 0.19$ is more modest and of the order of around 0.06% consumption equivalent compared with a weak rule of with feedback coefficients of 0.1 or an over-aggressive rule with feedback 0.5.

7.4 DYNARE CODES

The code for the material of this section is in the folder **Policy**. The mod file ***GK_equity_MPR.mod*** now has both a flexi-price bloc and an internal or exter-

feedback	Welfare	E	Spread	Y	C	I	H	R_n	SD(E)	SD(lever)
No MPR	-405.47	0.567	0.0044	0.766	0.474	0.146	0.324	1.0634	0.153	0.475
0.1	-406.07	0.555	0.0045	0.764	0.473	0.145	0.324	1.065	0.283	0.054
1.0	-406.60	0.249	0.0047	0.762	0.472	0.144	0.324	1.066	0.529	0.542

Table 7.1: A Regulatory Rule in the GK-equity Model.
The table reports ergodic means except where SDs are indicted. External Habit and Standard Taylor Monetary Rule. feedback=$lever_y = lever_{spread}$. $\rho_{lever} = 0.7$

feedback	Welfare	CE Cost of MPR	SD(lever)
No MPR	-405.47	0	0.475
0.1	-406.07	0.1276	0.054
0.17	-406.0623	0.1260	0.092
0.18	**-406.0622**	**0.1259**	0.095
0.19	**-406.0622**	**0.1259**	0.103
0.2	-406.06	0.1260	0.108
0.3	-406.07	0.1276	0.163
0.4	-406.10	0.1340	0.217
0.5	-406.14	0.1425	0.271
1.0	-406.60	0.2403	0.542

Table 7.2: Optimized Regulatory Rule in the GK-equity Model.
External Habit[6] and Standard Taylor Monetary Rule. feedback=$lever_y = lever_{spread}$. A 1% permanent increase in consumption gives a welfare gain of 4.7026

nal habit option. The exercise in Table 7.2 is carried out in ***GK_equity_MPR.mod*** with an option to turn off MPR and replace the rule with the bank's first-order condition for the decentralized choice of x_t, $(1+\lambda_t)\mu_{e,t} = \Theta'_t \lambda_t$.

7.5 EXERCISES

1. Use the graph plotter to compare of the GK model with and without MPR. In the former case choose the upper limit of the feedback parameter.

2. Rework Tables 7.1 and 7.2 with an implementable monetary rule. What do you notice?

8 BRIEF LITERATURE REVIEW

Prior to the financial crisis of 2007–2008, the literature on financial frictions in macroeconomics was relatively limited, and largely focused on asymmetric information problems and limited contract enforceability. Asymmetric information would emerge as either financing inefficiencies or co-ordination failures. For instance, Stiglitz and Weiss (1981) focus on the former and show how adverse selection in finance can lead to credit rationing in which some borrowers are excluded from the credit market at any price, even with profitable projects. Diamond and Dybvig (1983) is an example of the latter, highlighting how maturity mismatch with asymmetric information can lead to bank runs. The combination of short-term liabilities with only partially liquid long-term assets can generate a two-equilibrium model; a bank run equilibrium can occur as a self-fulfilling prophecy if households believe one might occur. In Bernanke and Gertler (1989), entrepreneurs observe information about their productivity and due to costly state verification (see Townsend (1979)) a wedge between the cost of internal and external finance that depends on the leverage of the borrower emerges, and leads to an endogenous default rate in equilibrium. This approach was extended further in Carlstrom and Fuerst (1997) and Bernanke *et al.* (1999). Holmstrom and Tirole (1997, 1998) discuss inefficient outcomes that emerge due to a dual moral hazard problem resulting from asymmetric information in financing projects. In this model, it is the responsibility of intermediaries to monitor the entrepreneurs, who are able to shirk. If intermediaries can also shirk, then there is a double moral hazard problem and both entrepreneur and intemediary will be capital constrained to ensure they both have sufficient 'skin in the game'.

Financial frictions emerging from limited commitment can take a similar form. For instance, collateral constraints arise in Kiyotaki and Moore (1997) but due to a commitment problem rather than asymmetric information; borrowers cannot commit to repay debt and so must hold collateral as a guarantee. This has an important effect on macroeconomic outcomes as durable goods take on the dual role of being both factors of production and sources of collateral. This dual role creates an accelerator mechanism as when the value of capital falls, firm net worth will also fall, tightening the credit constraint. The reverse is true as the credit constraint slackens during an upturn. Kehoe and Levine (1993) and Cooley *et al.* (2004) also look at limited contract enforceability and reach similar conclusions. The collateral constraints approach proposed in Kiyotaki and Moore (1997) was used to relate fluctuations in real estate prices with economic outcomes in Iacoviello (2005) by assuming that entrepreneurs must post real estate as collateral for loans, and by treating real estate as a factor of production. Here the accelerator mechanism of Kiyotaki and Moore (1997) worked via the housing market whereby a fall in house prices would both depress household demand and reduce investment.

Since the recent financial crisis, the number of papers studying the importance of financial frictions on macroeconomic outcomes and policy implications has grown considerably, commonly building on the mechanisms proposed in the Kiyotaki and Moore (1997) (KM) collateral constraints model, or the Bernanke *et al.* (1999) (BGG) costly state verification model. In KM the propagation and amplification comes from the fluctuations in asset prices, while in BGG from fluctuation of agents net worth. The KM approach was extended to study the effects of financial constraints on the banking sector in Gertler and Kiyotaki (2010) (GK) where the limited commitment problem of KM introduces an agency problem between depositors and banks; when the value of bank capital declines, the borrowing constraint tightens and limits the amount of deposits the bank can raise and subsequently, the level of investment. Another extension proposed in Gertler and Karadi (2011) uses this approach to analyse the role of unconventional monetary policy. It is assumed the central bank can perform financial intermediation at a cost, but when the borrowing constraint tightens sufficiently, this cost is less than the inefficiency introduced by the agency problem. The two approaches have both been applied to the housing market. Impatient households post housing as collateral to secure mortgage loans in Iacoviello and Neri (2010) where the mechanism of Iacoviello (2005) is focused on the demand-side of the economy, and shown to have important effects on the business cycle. The collateral constraints arise in Forlati and Lambertini (2011) due to the Bernanke *et al.* (1999) costly state verification mechanism which is applied to household credit by assuming households observe a private housing-value shock that can lead to default when households are insolvent. The authors emphasise increased housing investment risk in highly leveraged economies.

Of the alternative approaches to introduce credit frictions, Gerali *et al.* (2010) and Forni *et al.* (2010) introduce monopolistic competition into the banking sector with nominal interest rate rigidities. Kiyotaki and Moore (2012) and Adrian and Shin (2009) look at the role of liquidity; the former develop a model of monetary economy with differences in liquidity across assets, whilst the latter analyse how balance-sheet quantities of market-based financial intermediaries are important macroeconomic state variables for the conduct of monetary policy. Cúrdia and Woodford (2010) analyse the relationship between interest spreads and monetary policy by assuming that financial intermediation consumes real resources and that the credit spread depends on the volume of loans. Other papers have attempted to develop the bank-run model of Diamond and Dybvig (1983). Angeloni and Faia (2013) adapt Diamond (2000) and Diamond and Rajan (2000) (themselves a development of Diamond and Dybvig) to a DSGE setting, and Gertler and Kiyotaki (2012) incorporates both into a DSGE model.

The large influence of the BGG and KM approaches to the financial frictions literature might be partly due to the simplicity of applying the frictions to a representative agent, rational expectations model, solved using linear approximation techniques. It has been argued in Holden *et al.* (2019) that this rules out *ex ante* the possibility of explaining a number of key stylized facts, such as the large positive skew in the interest spreads, and negative skews in investment. There have been a number of papers that do study the non-linear and asymmetric effects of financial frictions which are much better suited to explain such phenomena, usually using global solution methods, or the perturbation based methods of Holden (2016) and Guerrieri and Iacoviello (2015). For example, Holden *et al.* (2019) extend the GK approach to allow for the borrowing constraint to only be occasionally binding. They find that the empirical accuracy of the simulated third moments and cross-correlations are significantly improved compared to a model with an always binding financial constraint. Other examples of financing constraints include He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014) and Dewachter and Wouters (2014) who propose an occasionally binding constraint on equity rather than debt. On the demand side, Iacoviello (2015) modifies the Iacoviello and Neri (2010) model by fixing the supply of housing and allowing the collateral constraint on household debt to be only occasionally binding. He shows how household mortgage default can lead to a credit crunch as banks de-leverage.

We now consider the general policy question: how when faced with the existence of multiple competing and contrasting models such as those studied on this financial frictions option, all of which are believed to be misspecified, should policymakers set macroeconomic policy? Deak *et al.* (2019) proposes general framework that uses a pool of contrasting models for the policy design problem. The methodology uses Bayesian estimation to weight alternative models to design optimized Taylor-type rules that are robust in a sense described below. A crucial requirement is to provide a k-period ahead predictive density, given macro-economic data. The predictive density characterizes out-of-sample observations that have not been used up to that point in time to estimate the posterior density of the parameter vector. As such this provides predictions about future observations that fully incorporate the information regarding within-model uncertainty (defined below) in the data.

The paper then investigates the welfare consequences of a standard Taylor-type monetary policy rules using three medium-scale New Keynesian DSGE models studied on this Course: the Smets-Wouters model in Smets and Wouters (2007a), the workhorse model widely used in policy-making institutions for forecasting and policy analysis, and the other two models that add GK and BGG variants of financial frictions. Hence, the model pool can be motivated by considering a policy maker who is uncertain how to incorporate financial frictions into a DSGE model or if they should be incorporated at all. For further details see the paper.

The purpose of this section was to provide some context for the models presented in the course, but does not constitute a comprehensive review of the financial frictions literature in macroeconomics. For a more comprehensive literature survey we refer to Brunnermeier *et al.* (2012). We also direct the interested reader to a recent special issue in the Review of Economic Dynamics in Gertler and Williamson (2015).

8.1 EXERCISE

As a take-away exercise from the Course add to the NK model a friction facing households in the housing market of the form set out in Iacoviello (2005). Compare the monetary policy transmission mechanism with that for the NK, GK and BGG models studied on this Course.

9 CONCLUSIONS

This one-day Course has covered two banking models suitable for incorporation into a DSGE modelling framework. We have shown how to set the models up in Dynare to perform second-order perturbation solutions, estimate the models using Bayesian methods and carry out optimal conventional and unconventional policy exercises.

Modelling financial frictions is a very active area of current research. Topics being investigated by the CIMS researchers include financial frictions in models with endogenous entry and medium term cycles, solutions to the case of occasionally binding constraints and international policy coordination. This Course has hopefully provided you with the tools necessary to participate in this exciting area.

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APPENDICES

1 THE HOUSEHOLD PROBLEM

Households own the capital stock which they rent to firms at a rental rate r_t^K. They choose between work and leisure and therefore how much labour they supply. Let $L_t = 1 - H_t$ be the total time available for work (say 16 hours per day) that consists of leisure time and H_t the proportion of this time spent at work. The single-period utility is

$$U = U(C_t, L_t) \tag{A.1}$$

and we assume that[1]

$$U_C > 0,\ U_L > 0\ U_{CC} \leq 0,\ U_{LL} \leq 0 \tag{A.2}$$

In a stochastic environment, the value function of the representative household at time t is given by

$$V_t = V_t(B_{t-1}) = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s U(C_{t+s}, L_{t+s}) \right] \tag{A.3}$$

The household's problem at time t is to choose paths for consumption $\{C_t\}$, leisure, $\{L_t\}$, labour supply $\{H_t = 1 - L_t\}$, capital stock $\{K_t\}$, investment $\{I_t\}$ and bond holdings to maximize V_t given by (A.3) given its budget constraint in period t

$$B_t = R_{t-1}B_{t-1} + W_t H_t - C_t - I_t - T_t \tag{A.4}$$

where B_t is the given net stock of financial assets at the end of period t, r_t^K is the rental rate, is the wage rate and R_t is the gross interest rate paid on assets held at the beginning of period t, I_t is investment and T_t are lump-sum taxes; and given that capital stock accumulates according to

$$K_t = (1-\delta)K_{t-1} + (1 - S(X_t))I_t\,; \tag{A.5}$$

$$X_t \equiv \frac{I_t}{I_{t-1}}\,;\ S',\, S'' \geq 0\,;\ S(1) = S'(1) = 0 \tag{A.6}$$

1 Our notation is $U_C \equiv \frac{\partial U}{\partial C}$, $U_{CC} \equiv \frac{\partial^2 U}{\partial C^2}$ etc.

In (A.6), $S(X_t)$ are investment adjustment costs, I_t units of output converts to $(1-S(X_t))I_t$ of new capital sold at a real price Q_t (Tobin's Q). All variables are expressed in real terms relative to the price of output.

To solve the household problem we form a Lagrangian

$$
\begin{aligned}
\mathcal{L} \;=\; & \mathbb{E}_t\Big[\sum_{s=0}^{\infty}\beta^s\Big(U(C_{t+s},L_{t+s}) \\
+\; & \lambda_{t+s}\Big[R_{t+s-1}B_{t+s-1}+W_{t+s}(1-L_{t+s})+r^K_{t+s}K_{t+s-1}-C_{t+s}-I_{t+s}-T_{t+s}-B_{t+s}] \\
+\; & Q_{t+s}[(1-\delta)K_{t+s-1}+(1-S(X_{t+s}))I_{t+s}-K_{t+s}]\Big)\Big]
\end{aligned}
$$

Then the first-order conditions with respect to $\{C_{t+s}\}$, $\{B_{t+s-1}\}$, $\{K_{t+s-1}\}$, $\{I_{t+s}\}$ and $\{L_{t+s}\}$ are respectively

$$\{C_{t+s}\} \;:\; \mathbb{E}_t[U_{C,t+s}-\lambda_{t+s}]=0\,; \quad s\geq 0 \tag{A.7}$$

$$\{B_{t+s-1}\} \;:\; \mathbb{E}_t[\beta^s\lambda_{t+s}R_{t+s-1}-\beta^{s-1}\lambda_{t+s-1}]=0\,; \quad s>0\ (B_{t-1}\text{ given}) \tag{A.8}$$

$$\{K_{t+s-1}\} \;:\; \mathbb{E}_t[\beta^s\lambda_{t+s}r^K_{t+s}+\beta^s\lambda_{t+s}Q_{t+s}(1-\delta)-\beta^{s-1}\lambda_{t+s-1}Q_{t+s-1}]=0\,; \quad s>0\ (K_{t-1}\text{ given}) \tag{A.9}$$

$$
\begin{aligned}
\{I_{t+s}\} \;:\; & \mathbb{E}_t\big[\lambda_{t,t+s}Q_{t+s}(1-S\left(I_{t+s}/I_{t+s-1}\right))-1-Q_{t+s}S'\left(I_{t+s}/I_{t+s-1}\right)\frac{I_{t+s}}{I_{t+s-1}} \\
- & \beta\lambda_{t,t+s+1}Q_{t+s+1}S'\left(I_{t+s}/I_{t+s-1}\right)\times\left(-\frac{I_{t+s+1}}{I^2_{t+s}}I_{t+s+1}\right)\big]=0\,; \quad s\geq 0
\end{aligned} \tag{A.10}
$$

$$\{L_{t+s}\} \;:\; \mathbb{E}_t[U_{L,t+s}-\lambda_{t+s}W_{t+s}]=0\,; \quad s\geq 0 \tag{A.11}$$

Putting $s=0$ in (A.7), (A.37) and (A.11) and $s=1$ in (A.8) and (A.9) and defining the stochastic discount factor as $\Lambda_{t,t+1}\equiv\beta\frac{\lambda_{t+1}}{\lambda_t}$ we now have:

$$\text{Euler Consumption } : 1 = R_t\mathbb{E}_t\left[\Lambda_{t,t+1}\right] \tag{A.12}$$

$$\text{Stochastic Discount Factor} : \Lambda_{t,t+1}\equiv\beta\frac{\lambda_{t+1}}{\lambda_t} \tag{A.13}$$

$$\text{where} : \lambda_t = U_{C,t} \tag{A.14}$$

$$\text{Labour Supply} : \frac{U_{H,t}}{\lambda_t} = -\frac{U_{L,t}}{\lambda_t} = -W_t \tag{A.15}$$

$$\text{Leisure and Hours}: L_t \equiv 1 - H_t \tag{A.16}$$

$$\begin{aligned}\text{Investment FOC}: &\; Q_t(1 - S(X_t) - X_t S'(X_t)) \\ &+ \mathbb{E}_t\left[\Lambda_{t,t+1}\, Q_{t+1} S'(\Xi_{t+1})\Xi_{t+1}^2\right] = 1\end{aligned} \tag{A.17}$$

$$\text{Capital Supply}: \mathbb{E}_t\left[\Lambda_{t,t+1} R_{t+1}^K\right] = 1 \tag{A.18}$$

where R_t^K is the gross return on capital given by

$$R_t^K = \frac{[r_t^K + (1-\delta)Q_t]}{Q_{t-1}} \tag{A.19}$$

1.1 Capital Producers

Note that the investment decision can be taken by separate capital producing firms who convert raw output into new capital at a cost and sold at a real price Q_t. They maximize with respect to $\{I_t\}$ expected discounted profits

$$\mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\left[Q_{t+s}(1 - S\left(I_{t+s}/I_{t+s-1}\right))I_{t+s} - I_{t+s}\right] \tag{A.20}$$

where $\Lambda_{t,t+s} = \beta^s \left(\frac{\lambda_{t+s}}{\lambda_t}\right)$ is the real stochastic discount rate over the interval $[t, t+s]$. This leads to the same first-order decision as (A.17).

1.2 Solution of the Household Problem with JR Preferences

Now form a Lagrangian

$$\begin{aligned}\mathcal{L} &= \mathbb{E}_t\Big[\sum_{s=0}^{\infty} \beta^s \Big(U(C_{t+s}, L_{t+s}, X_{t+s}) \\ &+ \lambda_{t+s}[R_{t+s-1}B_{t+s-1} + W_{t+s}(1 - L_{t+s}) + r_{t+s}^K K_{t+s-1} - C_{t+s} - I_{t+s} - T_{t+s} - B_{t+s}] \\ &+ \lambda_{t+s}Q_{t+s}[(1-\delta)K_{t+s-1} + (1 - S(X_{t+s}))I_{t+s} - K_{t+s}] + \mu_{t+s}[X_{t+s} - C_{t+s}^{\gamma} X_{t+s-1}^{1-\gamma}]\Big)\end{aligned}$$

Then the first-order conditions with respect to $\{X_{t+s}\}$, $\{C_{t+s}\}$, $\{B_{t+s-1}\}$, $\{K_{t+s-1}\}$, $\{I_{t+s}\}$ and $\{L_{t+s}\}$ are respectively

$$\{X_{t+s}\} \;:\; \mathbb{E}_t[U_{X,t+s} + \mu_{t+s} - \beta(1-\gamma)\mu_{t+s+1}C_{t+s+1}^{\gamma} X_{t+s}^{-\gamma}] = 0; \tag{A.21}$$

$$s \geq 0\{C_{t+s}\} \quad : \quad \mathbb{E}_t[U_{C,t+s} - \lambda_{t+s} - \gamma\mu_{t+s}C_{t+s-1}^{\gamma-1}X_{t+s-1}^{1-\gamma}] = 0\,;\; s \geq 0 \tag{A.22}$$

$$\{B_{t+s-1}\} \quad : \quad \mathbb{E}_t[\beta^s\lambda_{t+s}R_{t+s-1} - \beta^{s-1}\lambda_{t+s-1}] = 0\,;$$
$$s > 0\,(B_{t-1} \text{ given}) \tag{A.23}$$

$$\{K_{t+s-1}\} \quad : \quad \mathbb{E}_t[\beta^s\lambda_{t+s}r_{t+s}^K + \beta^s\lambda_{t+s}Q_{t+s}(1-\delta) - \beta^{s-1}\lambda_{t+s-1}Q_{t+s-1}] = 0\,;$$
$$s > 0\;(K_{t-1} \text{ given}) \tag{A.24}$$

$$\{I_{t+s}\} \quad : \quad \mathbb{E}_t\big[\lambda_{t,t+s}Q_{t+s}(1 - S\,(I_{t+s}/I_{t+s-1})) - 1 - Q_{t+s}S'\,(I_{t+s}/I_{t+s-1})\,\frac{I_{t+s}}{I_{t+s-1}}$$
$$- \quad \beta\lambda_{t,t+s+1}Q_{t+s+1}S'\,(I_{t+s}/I_{t+s-1}) \times \left(-\frac{I_{t+s+1}}{I_{t+s}^2}I_{t+s+1}\right)\big] = 0\,;$$
$$s \geq 0 \tag{A.25}$$

$$\{L_{t+s}\} \quad : \quad \mathbb{E}_t[U_{L,t+s} - \lambda_{t+s}W_{t+s}] = 0\,;\; s \geq 0 \tag{A.26}$$

Putting $s = 0$ in (A.21), (A.22), (A.25) and (A.26) and $s = 1$ in (A.23) and (A.24) and defining the stochastic discount factor as $\Lambda_{t,t+1} \equiv \beta\frac{\lambda_{t+1}}{\lambda_t}$ we now have:

$$\text{Euler Consumption} : 1 = R_t\mathbb{E}_t\left[\Lambda_{t,t+1}\right] \tag{A.27}$$

$$\text{Stochastic Discount Factor} : \Lambda_{t,t+1} \equiv \beta\frac{\lambda_{t+1}}{\lambda_t} \tag{A.28}$$

$$\text{where} : \lambda_t = U_{C,t} - \gamma\mu_t\frac{X_t}{C_t} \tag{A.29}$$

$$\text{and} : \mu_t = -U_{X,t} + \beta(1-\gamma)\mathbb{E}_t\frac{\mu_{t+1}X_{t+1}}{X_t} \tag{A.30}$$

$$\text{Labour Supply} : \frac{U_{H,t}}{\lambda_t} = -\frac{U_{L,t}}{\lambda_t} = -W_t \tag{A.31}$$

$$\text{Leisure and Hours} : L_t \equiv 1 - H_t \tag{A.32}$$

$$\text{Investment FOC} : Q_t(1 - S(\Xi_t) - \Xi_tS'(\Xi_t)) + \mathbb{E}_t\left[\Lambda_{t,t+1}\,Q_{t+1}S'(\Xi_{t+1})\Xi_{t+1}^2\right] = 1 \tag{A.33}$$

$$\text{Capital Supply} : \mathbb{E}_t\left[\Lambda_{t,t+1}R_{t+1}^K\right] = 1 \tag{A.34}$$

where R_t^K is the gross return on capital given by

$$R_t^K = \frac{\left[r_t^K + (1-\delta)Q_t\right]}{Q_{t-1}} \tag{A.35}$$

Investment Specific Shock

With an investment specific shock capital accumulation becomes

$$K_t = ((1-\delta)K_{t-1} + (1 - S(X_t))I_t IS_t) \tag{A.36}$$

Then the Lagrangian becomes

$$\begin{aligned}\mathcal{L} &= \mathbb{E}_t\Big[\sum_{s=0}^{\infty}\beta^s\Big(U(C_{t+s}, L_{t+s}, X_{t+s})\\ &+ \lambda_{t+s}[R_{t+s-1}B_{t+s-1} + W_{t+s}(1-L_{t+s}) + r^K_{t+s}K_{t+s-1} - C_{t+s} - I_{t+s} - T_{t+s} - B_{t+s}]\\ &+ \lambda_{t+s}Q_{t+s}[(1-\delta)K_{t+s-1} + (1-S(X_{t+s}))I_{t+s}IS_{t+1}] - K_{t+s} + \mu_{t+s}[X_{t+s} - C^{\gamma}_{t+s}X^{1-\gamma}_{t+s-1}]\Big)\end{aligned}$$

Then the first-order conditions with respect to $\{I_{t+s}\}$ become

$$\begin{aligned}\{I_{t+s}\} \quad &: \quad \mathbb{E}_t\big[\lambda_{t,t+s}Q_{t+s}IS_{t+s}(1 - S\left(I_{t+s}/I_{t+s-1}\right)) - 1 - Q_{t+s}IS_{t+s}S'\left(I_{t+s}/I_{t+s-1}\right)\frac{I_{t+s}}{I_{t+s-1}}\\ &- \quad \beta\lambda_{t,t+s+1}Q_{t+s+1}IS_{t+s+1}S'\left(I_{t+s}/I_{t+s-1}\right) \times \left(-\frac{I_{t+s+1}}{I^2_{t+s}}I_{t+s+1}\right)\big] = 0\,;\\ s \geq 0\end{aligned} \tag{A.37}$$

leading to changes

$$\text{Investment FOC}: Q_t IS_t(1 - S(\Xi_t) - \Xi_t S'(\Xi_t)) + \mathbb{E}_t\left[\Lambda_{t,t+1}\,Q_{t+1}IS_{t+1}S'(\Xi_{t+1})\Xi^2_{t+1}\right] = 1 \tag{A.38}$$

$$\text{Capital Supply}: \mathbb{E}_t\left[\Lambda_{t,t+1}R^K_{t+1}\right] = 1 \tag{A.39}$$

where R^K_t is the gross return on capital given by

$$R^K_t = \frac{\left[r^K_t + (1-\delta)Q_t\right]}{Q_{t-1}} \tag{A.40}$$

2 INFLATION AND PRICE DISPERSION DYNAMICS

This Appendix shows how first order conditions expressed as summations, as in Calvo contracts, can be expressed as difference equations suitable for coding in Dynare. Then the dynamic form of price dispersion, Δ_t is derived.

2.1 A Useful Lemma

In the first order conditions for Calvo contracts and expressions for value functions we are confronted with expected discounted sums of the general form

$$\Omega_t = \mathbb{E}_t \left[\sum_{k=0}^{\infty} \beta^k X_{t,t+k} Y_{t+k} \right] \tag{B.1}$$

where $X_{t,t+k}$ has the property $X_{t,t+k} = X_{t,t+1} X_{t+1,t+k}$ and $X_{t,t} = 1$ (for example an inflation, interest or discount rate over the interval $[t, t+k]$).

Lemma

Ω_t can be expressed as

$$\Omega_t = Y_t + \beta \mathbb{E}_t \left[X_{t,t+1} \Omega_{t+1} \right] \tag{B.2}$$

Proof

$$\begin{aligned}
\Omega_t &= X_{t,t} Y_t + \mathbb{E}_t \left[\sum_{k=1}^{\infty} \beta^k X_{t,t+k} Y_{t+k} \right] \\
&= Y_t + \mathbb{E}_t \left[\sum_{k'=0}^{\infty} \beta^{k'+1} X_{t,t+k'+1} Y_{t+k'+1} \right] \\
&= Y_t + \beta \mathbb{E}_t \left[\sum_{k'=0}^{\infty} \beta^{k'} X_{t,t+1} X_{t+1,t+k'+1} Y_{t+k'+1} \right] \\
&= Y_t + \beta \mathbb{E}_t \left[X_{t,t+1} \Omega_{t+1} \right] \qquad \square
\end{aligned}$$

2.2 Price Dynamics

Then (2.9) in the main text can be written

$$\frac{P_t^0}{P_t} = \frac{J_t^p}{JJ_t^p} \tag{B.3}$$

and summations JJ_t^p and J_t^p are of the form considered in the Lemma above. Applying the Lemma, inflation dynamics are given by

$$\begin{aligned}
JJ_t^p - \xi \mathbb{E}_t [\Lambda_{t,t+1} \Pi_{t+1}^{\zeta-1} JJ_{t+1}^p] &= Y_t \\
J_t^p - \xi \mathbb{E}_t [\Lambda_{t,t+1} \Pi_{t+1}^{\zeta} J_{t+1}^p] &= \frac{1}{1-\frac{1}{\zeta}} Y_t MC_t MS_t
\end{aligned}$$

$$
\begin{aligned}
1 &= \xi \Pi_t^{\zeta-1} + (1-\xi)\left(\frac{J_t^p}{JJ_t^p}\right)^{1-\zeta} \\
MC_t &= \frac{P_t^W}{P_t} = \frac{W_t}{P_t F_{H,t}} \\
\Delta_t &\equiv \frac{1}{n}\sum_{j=1}^{n}(P_t(j)/P_t)^{-\zeta} = \xi \Pi_t^{\zeta}\Delta_{t-1} + (1-\xi)\left(\frac{J_t^p}{JJ_t^p}\right)^{-\zeta}
\end{aligned}
\tag{B.4}
$$

as in the main text.

With *indexing* by an amount $\gamma \in [0,1]$, the optimal price-setting first-order condition for a firm j setting a new optimized price $P_t^0(j)$ is now given by

$$
P_t^0(j) = \frac{\frac{1}{(1-1/\zeta)}\mathbb{E}_t\left[\sum_{k=0}^{\infty}\xi^k \Lambda_{t,t+k} P_{t+k} MC_{t+k} MS_{t+k} Y_{t+k}(j)\right]}{\mathbb{E}_t\left[\sum_{k=0}^{\infty}\xi^k D_{t,t+k} Y_{t+k}(j)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma}\right]}
$$

Price dynamics are now given by

$$
\begin{aligned}
\frac{P_t^0}{P_t} &= \frac{J_t^p}{JJ_t^p} \\
JJ_t^p - \xi\beta\mathbb{E}_t[\tilde{\Pi}_{t+1}^{\zeta-1} JJ_{t+1}^p] &= Y_t U_{C,t} \\
J_t^p - \xi\beta\mathbb{E}_t[\tilde{\Pi}_{t+1}^{\zeta} J_{t+1}^p] &= \frac{1}{1-\frac{1}{\zeta}} MC_t MS_t Y_t U_{C,t} \\
\tilde{\Pi}_t &\equiv \frac{\Pi_t}{\Pi_{t-1}^{\gamma}}
\end{aligned}
$$

2.3 Dynamics of Price Dispersion

Price dispersion lowers aggregate output as follows. As with consumption goods, the demand equations for each differentiated good m with price $P_t(m)$ forming aggregate investment and public services takes the form

$$
I_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} I_t\,; \quad G_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} G_t \tag{B.5}
$$

Hence equilibrium for good m gives

$$
Y_t(m) = A_t H_t(m)\left(\frac{K_t(m)}{Y_t(m)}\right)^{\frac{1-\alpha}{\alpha}} = (C_t + I_t + G_t)\left(\frac{P_t(m)}{P_t}\right)^{-\zeta} \tag{B.6}
$$

where $Y_t(m)$, $H_t(m)$ and $K_t(m)$ are the quantities of output, hours and capital needed in the wholesale sector to produce good m in the retail sector. Since the capital-labour ratio is constant integrating over m, and using $H_t = \int_0^1 H_t(m)dm$ we obtain

$$Y_t = \frac{F(A_t, H_t, K_t)}{\Delta_t^p} \tag{B.7}$$

as in the main text.

Price dispersion is linked to inflation as follows. Assuming as before that the number of firms is large we obtain the following dynamic relationship:

$$\Delta_t^p = \xi \Pi_t^{\zeta} \Delta_{t-1}^p + (1-\xi)\left(\frac{J_t^p}{JJ_t^p}\right)^{-\zeta} \tag{B.8}$$

Proof

In the next period, ξ of these firms will keep their old prices, and $(1-\xi)$ will change their prices to P_{t+1}^O. By the law of large numbers, we assume that the distribution of prices among those firms that do not change their prices is the same as the overall distribution in period t. It follows that we may write

$$\begin{aligned}
\Delta_{t+1}^p &= \xi \sum_{j_{no\ change}} \left(\frac{P_t(j)}{P_{t+1}}\right)^{-\zeta} + (1-\xi)\left(\frac{J_{t+1}^p}{JJ_{t+1}^p}\right)^{-\zeta} \\
&= \xi \left(\frac{P_t}{P_{t+1}}\right)^{-\zeta} \sum_{j_{no\ change}} \left(\frac{P_t(j)}{P_t}\right)^{-\zeta} + (1-\xi)\left(\frac{J_{t+1}^p}{JJ_{t+1}^p}\right)^{-\zeta} \\
&= \xi \left(\frac{P_t}{P_{t+1}}\right)^{-\zeta} \sum_{j} \left(\frac{P_t(j)}{P_t}\right)^{-\zeta} + (1-\xi)\left(\frac{J_{t+1}^p}{JJ_{t+1}^p}\right)^{-\zeta} \\
&= \xi \Pi_{t+1}^{\zeta} \Delta_t^p + (1-\xi)\left(\frac{J_{t+1}^p}{JJ_{t+1}^p}\right)^{-\zeta} \qquad \square
\end{aligned}$$

Wage stickiness follows in an analogous fashion.

3 ORIGINAL GK SOLUTION OF THE GK MODEL

3.1 GK with Internal Equity Only

To solve this problem we first note using (3.1) that $V_t(s_t, d_t)$ is homogeneous of degree one in s_t and d_t. The solution must therefore take the form:

$$V_t = V_t(s_t, d_t) = \nu_{s,t}s_t - \nu_{d,t}d_t \tag{C.1}$$

where $\nu_{s,t}$, and $\nu_{d,t}$ are time-varying parameters that are the marginal values of the asset at the end of period t. Now eliminate d_t from (C.1) using (3.1) to obtain

$$V_t = V_t(s_t, n_t) = \mu_{s,t}Q_t s_t + \nu_{d,t}n_t \tag{C.2}$$

where $\mu_{s,t} \equiv \frac{\nu_{s,t}}{Q_t} - \nu_{d,t}$ is the excess value of bank assets over deposits.

Next write the Bellman equation as

$$V_t(s_t, n_t) = \max_{s_t} \mathbb{E}_t\Lambda_{t,t+1}[(1-\sigma_B)n_{t+1} + \sigma_B V_{t+1}(s_{t+1}, n_{t+1})] \tag{C.3}$$

with n_t given by (3.2). Then we perform the optimization $\max_{s_t} V_t(s_t, n_t)$ subject to the IC constraint (3.5). The Lagrangian for this problem is

$$\mathcal{L}_t \;=\; V_t + \lambda_t[V_t - \Theta Q_t s_t] = (1+\lambda_t)V_t - \lambda_t\Theta Q_t s_t \tag{C.4}$$

where $\lambda_t > 0$ if the constraint binds and $\lambda_t = 0$ otherwise.

The first order and Kuhn-Tucker conditions for the optimization problem are:

$$s_t \;:\; (1+\lambda_t)\mu_{s,t} = \lambda_t\Theta \tag{C.5}$$

$$KT \;:\; \lambda_t(\mu_{s,t}Q_t s_t + \nu_{d,t}n_t - \Theta Q_t s_t) = 0 \tag{C.6}$$

$$KT \;:\; \lambda_t \geq 0 \tag{C.7}$$

We now define ϕ_t as the leverage ratio (loans to net worth) of the representative bank

$$\phi_t \equiv \frac{Q_t s_t}{n_t} \tag{C.8}$$

Then if $\lambda_t > 0$ and the IC constraint binds, using (C.5), ϕ_t is given by

$$\phi_t = \frac{\nu_{d,t}}{\Theta - \mu_{s,t}} \tag{C.9}$$

whereas if it does not bind, from (C.5) we have $\mu_{s,t} = 0$. Taking these together we can write

$$\mu_{s,t} = \max\left\{0,\ \Theta - \frac{\nu_{d,t}}{\phi_t}\right\} \tag{C.10}$$

Using (C.8) we can write (C.2) as

$$V_t = [\mu_{s,t}\phi_t + \nu_{d,t}]n_t \tag{C.11}$$

and hence (C.3) becomes

$$\begin{aligned} V_t(s_t, n_t) &= \mathbb{E}_t\Lambda_{t,t+1}[1 - \sigma_B + \sigma_B(\mu_{s,t+1}\phi_{t+1} + \nu_{d,t+1})]n_{t+1} \\ &\equiv \mathbb{E}_t\Lambda_{t,t+1}\Omega_{t+1}n_{t+1} \\ &= \mathbb{E}_t\Lambda_{t,t+1}\Omega_{t+1}[R^K_{t+1}Q_t s_t - R_{t+1}d_t] \end{aligned} \tag{C.12}$$

defining $\Omega_t = 1 - \sigma_B + \sigma_B(\nu_{d,t} + \phi_t\mu_{s,t}) = 1 - \sigma_B + \sigma_B\Theta\phi_t$ if the IC constraint binds, the shadow value of a unit of net worth (using (C.9)).

Comparing (C.12) with (C.1) and equating coefficients of s_t and d_t, we arrive at the determination of $\nu_{s,t}$ and $\nu_{d,t}$:

$$\nu_{d,t} = \mathbb{E}_t\Lambda_{t,t+1}\Omega_{t+1}R_{t+1} \tag{C.13}$$

$$\nu_{s,t} = \mathbb{E}_t\Lambda_{t,t+1}\Omega_{t+1}Q_t R^K_{t+1} \tag{C.14}$$

Hence we have the excess value of bank assets over deposits

$$\mu_{s,t} \equiv \frac{\nu_{s,t}}{Q_t} - \nu_{d,t} = \mathbb{E}_t\Lambda_{t,t+1}\Omega_{t+1}(R^K_{t+1} - R_{t+1}) \tag{C.15}$$

Equations (C.10) and (C.13)–(C.15) complete the bankers solution and determine ϕ_t, $\mu_{s,t}$, $\nu_{d,t}$ and $\nu_{s,t}$ in terms of the interest rate wedge $R^K_t - R_t$. Note that in the absence of a binding IC constraint, $\mu_{s,t} = 0$, $\Omega_t = 1$, $\nu_{d,t} = \mathbb{E}_t\Lambda_{t,t+1}R_{t+1} = \mathbb{E}_t\Lambda_{t,t+1}R^K_{t+1} = 1$, the arbitrage condition for our NK model, and leverage is indeterminate.

3.2 GK with Outside Equity

Proceeding as before, to solve the problem we look for a linear solution of the form:

$$V_t = V_t(s_t, d_t, e_t) = \nu_{s,t}s_t - \nu_{d,t}d_t - \nu_{e,t}e_t \tag{C.16}$$

where $\nu_{s,t}/Q_t$, $\nu_{d,t}$, $\nu_{e,t}/q_t$ are time-varying parameters that are the marginal values of the asset at the end of period t. Now eliminate d_t from (C.16) using (3.15) to obtain

$$V_t = V_t(s_t, e_t, n_t) = \mu_{s,t}Q_t s_t + \mu_{e,t}q_t e_t + \nu_{d,t}n_t = (\mu_{s,t} + \mu_{e,t}x_t)Q_t s_t + \nu_{d,t}n_t \tag{C.17}$$

where $\mu_{s,t} \equiv \frac{\nu_{s,t}}{Q_t} - \nu_{d,t}$ is the excess value of bank assets over deposits and $\mu_{e,t} \equiv \nu_{d,t} - \frac{\nu_{e,t}}{q_t}$ is the excess cost of deposits over outside equity.

Using (C.17) we can write the Bellman equation as

$$V_t(s_t, x_t, n_t) = \max_{s_t, x_t} \mathbb{E}_t \Lambda_{t,t+1}[(1-\sigma_B)n_{t+1} + \sigma_B V_{t+1}(s_{t+1}, x_{t+1}, n_{t+1})] \quad \text{(C.18)}$$

Then we perform the optimization $\max_{s_t,x_t} V_t(s_t, x_t, n_t)$ subject to the IC constraint (3.18). The Lagrangian for this problem is

$$\mathcal{L}_t = V_t + \lambda_t[V_t - \Theta(x_t)Q_t s_t] = (1+\lambda_t)V_t - \lambda_t \Theta(x_t) Q_t s_t \quad \text{(C.19)}$$

where $\lambda_t > 0$ if the constraint binds and $\lambda_t = 0$ otherwise.

The first order conditions for the optimization problem are:

$$\begin{aligned} s_t &: (1+\lambda_t)(\mu_{s,t} + \mu_{e,t}x_t) = \lambda_t \Theta(x_t) \\ x_t &: (1+\lambda_t)\mu_{e,t} = \lambda_t \Theta'(x_t) \\ KT &: \lambda_t(\mu_{s,t} + \mu_{e,t}x_t)Q_t s_t + \nu_{d,t}n_t - \Theta(x_t)Q_t s_t) = 0 \\ KT &: \lambda_t \geq 0 \end{aligned}$$

As before we define ϕ_t be the leverage ratio of the representative bank

$$\phi_t = \frac{Q_t s_t}{N_t} \quad \text{(C.20)}$$

where if $\lambda_t > 0$ and the IC constraint binds ϕ_t is given by

$$\phi_t = \frac{\nu_{d,t}}{\Theta_t - (\mu_{s,t} + \mu_{e,t}x_t)} \quad \text{(C.21)}$$

Otherwise if it does not bind $\lambda_t = 0$ so from the first order conditions $\mu_{e,t} = \mu_{s,t} = 0$. Thus we can write

$$\mu_{s,t} = \max\left\{0,\ \Theta_t - \mu_{e,t}x_t - \frac{\nu_{d,t}}{\phi_t}\right\} \quad \text{(C.22)}$$

Using (C.20) we can write (C.2) as

$$V_t = [(\mu_{s,t} + \mu_{e,t}x_t)\phi_t + \nu_{d,t}]n_t \quad \text{(C.23)}$$

and hence (C.18) becomes

$$\begin{aligned} V_t(s_t, e_t, n_t) &= \mathbb{E}_t \Lambda_{t,t+1}[1 - \sigma_B + \sigma_B((\mu_{s,t+1} + \mu_{e,t+1}x_{t+1})\phi_{t+1} + \nu_{d,t+1})]n_{t+1} \\ &\equiv \mathbb{E}_t \Lambda_{t,t+1} \Omega_{t+1} n_{t+1} \end{aligned}$$

$$= \mathbb{E}_t \Lambda_{t,t+1} \Omega_{t+1} [R^K_{t+1} Q_t s_t - R_{t+1} d_t - R^E_{t+1} q_t e_t] \tag{C.24}$$

defining $\Omega_t = 1 - \sigma_B + \sigma_B(\nu_{d,t} + \phi_t(\mu_{s,t} + \mu_{e,t} x_t))$, the shadow value of a unit of net worth, and using (3.16).

Comparing (C.24) with (C.16) and equating coefficients of s_t, d_t and e_t we arrive at the determination of $\nu_{s,t}$, $\nu_{d,t}$ and $\nu_{e,t}$:

$$\nu_{d,t} = \mathbb{E}_t \Lambda_{t,t+1} \Omega_{t+1} R_{t+1} \tag{C.25}$$

$$\nu_{s,t} = \mathbb{E}_t \Lambda_{t,t+1} \Omega_{t+1} Q_t R^K_{t+1} \tag{C.26}$$

$$\nu_{e,t} = \mathbb{E}_t \Lambda_{t,t+1} \Omega_{t+1} q_t R^E_{t+1} \tag{C.27}$$

Hence

$$\mu_{s,t} \equiv \frac{\nu_{s,t}}{Q_t} - \nu_{d,t} = \mathbb{E}_t \Lambda_{t,t+1} \Omega_{t+1} (R^K_{t+1} - R_{t+1}) \tag{C.28}$$

$$\mu_{e,t} \equiv \nu_{d,t} - \frac{\nu_{e,t}}{q_t} = \mathbb{E}_t \Lambda_{t,t+1} \Omega_{t+1} (R_{t+1} - R^E_{t+1}) \tag{C.29}$$

Equations (C.22), and (C.25) – (C.29) complete the bankers solution and determine ϕ_t, $\mu_{s,t}$, $\mu_{e,t}$, $\nu_{d,t}$, $\nu_{s,t}$ and $\nu_{e,t}$ in terms of the interest rate wedges $R^K_t - R_t$ and $R_t - R^E_t$.