The Science and Art of DSGE Modelling
A Foundations Course

Solution, Determinacy and Stability

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Dynare Set-up and Solution Method

- Our models are all special cases of the following general setup recognized by Dynare in non-linear form

$$Z_t = J(Z_{t-1}, X_t) + \nu\epsilon_{t+1} \quad (1)$$

$$\mathbb{E}_t X_{t+1} = K(Z_t, X_t) \quad (2)$$

where Z_{t-1}, X_t are $(n-m)\times 1$ and $m \times 1$ vectors of backward and forward-looking variables, respectively, and ϵ_t is a $\ell \times 1$ shock variable and ν is an $(n-m)\times \ell$ matrix.

- The two methods Dynare uses to solve models are perturbation methods for **stochastic simulations** when the command *stoch_simul* is used and Newton-type methods for **deterministic simulations** (perfect foresight) when the command *simul* is used.
- On the Course we only use the former.
- But the perfect foresight solution has the advantage it is *exact* and applies to *large* deviations about a steady state.

Dynare Set-up and Solution Method

- To see how this solution method works define

$$y_t \equiv \begin{bmatrix} Z_t \\ X_t \end{bmatrix} \tag{3}$$

- Then, as in Dynare User Guide, (1) and (2) can be written

$$\begin{aligned} \mathbb{E}_t[f(y_{t+1}, y_t, y_{t-1}, \epsilon_t)] &= 0 \\ \mathbb{E}_{t-1}[\epsilon_t] &= 0 \\ \mathbb{E}_{t-1}[\epsilon_t \epsilon_t'] &= \Sigma_\epsilon \end{aligned} \tag{4}$$

- Solution in Dynare is by *Undetermined Coefficients*. We look for a solution of the form

$$y_t = g(y_{t-1}, \epsilon_t) = 0 \tag{5}$$

called the *policy or decision function*. Then we have $y_{t+1} = g(y_t, \epsilon_{t+1}) = g(g(y_{t-1}, \epsilon_t), \epsilon_{t+1})$. Hence (4) can be rewritten:

$$\mathbb{E}_t[f(y_{t+1}, y_t, y_{t-1}, \epsilon_{t+1})] = \mathbb{E}_t[F(y_{t-1}, \epsilon_t, \epsilon_{t+1})] = 0 \tag{6}$$

Dynare Set-up and Solution Method

- Now expand $F(\cdot)$ as a *Taylor series expansion* about a deterministic (possibly trended) steady state defined from (4) by

$$f(\mathrm{y}, \mathrm{y}, \mathrm{y}, 0) = 0 \tag{7}$$

- Define a deviation $\hat{\mathrm{y}}_t \equiv \mathrm{y}_t - \mathrm{y}$ and denote partial derivatives $f_{y+} \equiv \frac{\partial f}{\partial y_{t+1}}$, $g_y \equiv \frac{\partial g}{y_{t-1}}$ etc evaluated at the steady state.
- Then using $\mathbb{E}_t[\epsilon_{t+1}] = 0$, the *first order Taylor Series expansion* of $F(\cdot)$ yields

$$(f_{y+}g_y g_y + f_y g_y + f_{y-})\hat{\mathrm{y}}_{t-1} + (f_{y+}g_y g_\epsilon + f_y g_\epsilon + f_\epsilon)\epsilon_t = 0 \tag{8}$$

- Now the crucial step! Since (8) must hold for any random shock, each term in the brackets $(\cdot)$ above must be equal to zero. Therefore can solve for g_y and g_ϵ yielding the policy function in linearized form

$$\mathrm{y}_t = \bar{\mathrm{y}} + g_y \hat{\mathrm{y}}_{t-1} + g_\epsilon \epsilon_t \tag{9}$$

Dynare Set-up and Solution Method

- We have now obtained the *first-order perturbation method solution*.
- Second and higher order approximations can be obtained by higher order Taylor series and are available in Dynare up to order 3. Default is order 2.
- When you linearize by hand and solve you arrive at the first-order perturbation solution from the non-linear model
- The Blanchard-Kahn minimal state-space form considered next can be obtained from the general set-up using the algorithm of Levine and Pearlman (2011).
- The following treatment of determinacy and stability complements that of the Global Sensitivity Analysis (GSA).

Determinacy and Stability

Using Levine and Pearlman (2011), we can express linearized forms of the models in Blanchard-Kahn form:

$$\begin{bmatrix} z_{t+1} \\ \mathbb{E}_t x_{t+1} \end{bmatrix} = A \begin{bmatrix} z_t \\ x_t \end{bmatrix} + Bw_t + C\epsilon_{t+1} \, ; \quad o_t = E \begin{bmatrix} z_t \\ x_t \end{bmatrix} \tag{10}$$

- where z_t is a $(n-m)\times 1$ vector of predetermined variables at time t with z_0 given, x_t, is a $m \times 1$ vector of non-predetermined variables and o_t is a vector of outputs.
- w_t is a vector of instruments (such as nominal interest rate, tax rates etc) exogenous for now.
- All variables are expressed as absolute or proportional deviations about a steady state.
- A, B, C and E are fixed matrices and ϵ_t as a vector of random zero-mean shocks.

Determinacy and Stability

- If we substitute into (10) a simple rule of the form

$$\mathrm{w} = D\mathrm{y}_t = D \begin{bmatrix} \mathrm{z}_t \\ \mathrm{x}_t \end{bmatrix} \tag{11}$$

- This rule is quite general as the state vector can be augmented to include lags (see Taylor rule in the NK model later).
- The condition for a stable and unique equilibrium depends on the magnitude of the eigenvalues of the matrix $A + BD$.
- If the number of eigenvalues outside the unit circle is equal to the number of non-predetermined variables, the system has *a unique equilibrium* which is also stable with saddle-path $x_t = -Nz_t$ where $N = N(D)$. (See Blanchard and Kahn (1980); Currie and Levine (1993)).

Determinacy and Stability

- In our NK model there are 5 non-predetermined (or jumpers) variables with forward looking expectation: π_t, q_t, i_t and c_t and the expected real interest rate.
- **Instability** occurs when the number of eigenvalues of $A + BD$ outside the unit circle is larger than the number of non-predetermined variables. (*This implies that when the economy is pushed off its steady state following a shock, it cannot ever converge back to it, but rather finishes up with explosive dynamics.*)
- **Indeterminacy** occurs when the number of eigenvalues of $A + BD$ outside the unit circle is smaller than the number of non-predetermined variables. (*This implies that when a shock displaces the economy from its steady state, there are many possible paths leading back to equilibrium, i.e. there are multiple stable rational expectations solutions to the model economy.*)

Dynare and Matlab Files

- Dynare files and a graph plotter Matlab file to run the linearized RBC model and compare it with the non-linear set-up are in the sub-folder **RBC linear**.
- The folder **Determinacy-Stability** provides the linearized NK model and checks the stability properties in the parameter space of (ρ_r, θ_π) using Matlab file **indeter.m**.
- **indeter.m**: loops a .mod file that reports and plots the results of the Blanchard-Kahn determinacy and stability condition for a given model.
- The Matlab file provided here calls the NK model which is adapted to specify the parameter space of (ρ_r, θ_π) and examines the Blanchard-Kahn condition for different values of coefficients in the Taylor rule
- Can be easily adapted to any model.

Dynare Exercises

1. Rework the stability analysis with a one-period ahead inflation targeting rule ($j = 1$).
2. Extend this case to $j = 2, 3, 4$. What do you notice?

Blanchard, O. J. and Kahn, C. M. (1980). The Solution of Linear Difference Models Under Rational Expectations. *Econometrica*, **48**(5), 1305–11.

Currie, D. and Levine, P. (1993). *Rules, Reputation and Macroeconomic Policy Coordination*. Cambridge University Press.

Levine, P. and Pearlman, J. (2011). Computation of LQ Approximations to Optimal Policy Problems in Different Information Settings under Zero Lower Bound Constraints. Dynare Working Papers 10, CEPREMAP.