The Science and Art of DSGE Modelling
A Foundations Course

The New Keynesian Model

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From RBC to NK

Recall The RBC Core

- Households, with external habit in their utility, make an intertemporal utility-maximizing choice of consumption and labour supply subject to a budget constraint.
- Net assets consists of capital employed by firms
- Firms (wholesale=retail) produce output according to a crt production technology and choose labour and capital inputs to minimize cost
- Investment adjustment costs and capital producers
- Labour, output and financial markets clear
- A balanced exogenous growth steady state with *zero growth*

Three NK Models

- Recall the three models from the Course Introduction

  1. The "workhorse" NK model with labour as the only factor of production. Prices are sticky, but wages flexible. Suitable for analytical results - see Woodford (2003) for example.
  2. The model of Smets and Wouters (2007), henceforth SW with capital, sticky wages, capacity utilization and fixed costs.
  3. A "slimmed down" SW model without sticky wages, capacity utilization and fixed costs.

- Model 3 is a special case of Model 2 and Model 1 is a special case of Model 3 (and therefore Model 2).
- We first concentrate on models 3 which is the model we estimate. Then we cover models 1 and 2.

The New Keynesian Models 1 and 3

- Our NK **model 3** consists of
  - A RBC real core with CD utility, labour and capital as factors of production, external habit in consumption and a nominal side consisting of
  - prices set by the retail sector
  - a nominal interest rate set by the policymaker
- We introduce *price stickiness* in the form of staggered Calvo-type price setting in the retail sector
- The literature often presents a *3-equation* log-linearized form of **model 1**, the workhorse model, with logarithmic utility, no habit, labour as the only factor of production and no government expenditure ($\sigma_c = 1$, $\chi = I_t = G_t = 0$).
- But this is our Model 1 considered later as a special case of Model 3

Illustration of NK Model

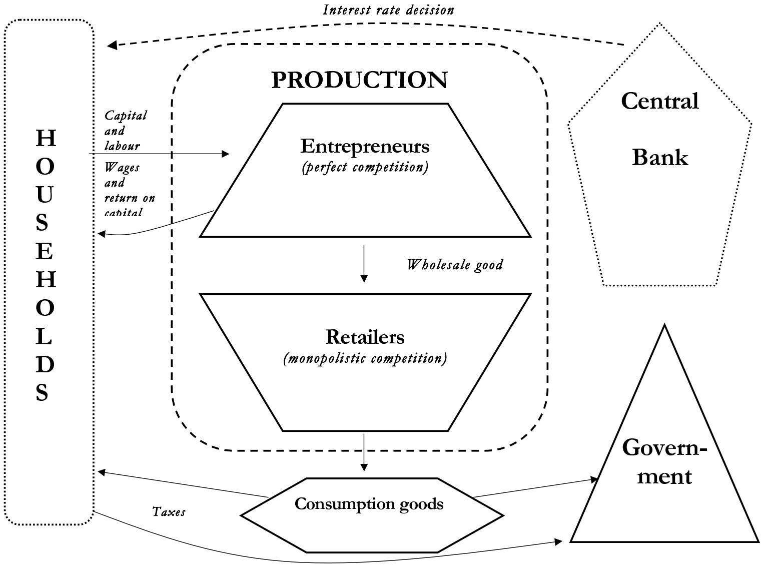

Figure: **NK Model**

The NK Model 3: Dixit-Stiglitz Aggregators

- Following Dixit and Stiglitz (1977) assume *monopolistic competition* in the retail sector that converts a homogeneous wholesale good into differentiated goods.
- See Appendix 2 of the Notes for a summary of this seminal paper.
- Then we can model *price-setting* by firms.
- Define *Dixit-Stiglitz CES aggregators* for aggregate consumption and price over differentiated goods m by

$$C_t = \left(\int_0^1 C_t(m)^{(\zeta-1)/\zeta} dm \right)^{\zeta/(\zeta-1)}$$

$$P_t = \left[\int_0^1 P_t(m)^{1-\zeta} dm \right]^{\frac{1}{1-\zeta}}$$

- With these definitions we have that

$$P_t C_t = \text{aggregate expenditure}$$

- The parameter ζ turns out to be a *price-elasticity*.

The NK Model 3: Demand for Differentiated Goods

- From Appendix 2 of the Notes, Dixit-Stiglitz aggregators lead to a demand for consumption of good m is given by

$$C_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} C_t$$

- Similarly for investment and government goods so in aggregate

$$Y_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} Y_t \tag{1}$$

where $Y_t(m)$ is the quantities of output needed in the wholesale sector to produce good m in the retail sector.

- Integrating over m we then have

$$\int_0^1 Y_t(m)dm = Y_t^W = \left(\int_0^1 \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} dm\right) Y_t = \Delta_t Y_t \tag{2}$$

where $\Delta_t \equiv \int_0^1 \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} dm$ is *price dispersion*.

The NK Model 3: Calvo Price Contracts

- Now we can model *price stickiness*
- There is a probability of $1-\xi$ at each period that the price of each retail good m is set optimally to $P_t^0(m)$; otherwise it is held fixed.
- Retail producer m, given the common real marginal cost $MC_t(m) = MC_t$ chooses $\{P_t^0(m)\}$ to maximize discounted real profits

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \frac{\Lambda_{t,t+k}}{P_{t+k}} Y_{t+k}(m) \left[P_t^0(m) - P_{t+k} MC_{t+k}\right]$$

where $\Lambda_{t,t+k} \equiv \beta^k \frac{U_{C,t+k}}{U_{C,t}}$ is the *stochastic discount factor* over the interval $[t, t+k]$, subject to

$$Y_{t+k}(m) = \left(\frac{P_t^O(m)}{P_{t+k}}\right)^{-\zeta} Y_{t+k}$$

(using (1)).

The NK Model 3: Calvo Price Contracts

- The solution to this optimization problem is

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \frac{\Lambda_{t,t+k}}{P_{t+k}} Y_{t+k}(m) \left[P_t^0(m) - \frac{1}{(1-1/\zeta)} P_{t+k} MC_{t+k}(m) \right] = 0$$

- Hence

$$\frac{P_t^O(m)}{P_t} = \frac{1}{(1-1/\zeta)} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \Lambda_{t,t+k} \left(\Pi_{t,t+k}\right)^{\zeta} Y_{t+k} MC_{t+k}(m)}{\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \Lambda_{t,t+k} \left(\Pi_{t,t+k}\right)^{\zeta-1} Y_{t+k}(m)}$$

- In a symmetric equilibrium of identical costs $MC_t(m) = MC_t$; hence $P_t^O(m) = P_t^O$ and $Y_t(m) = Y_t$ are invariant across m
- Hence we can write

$$\frac{P_t^0}{P_t} = \frac{J_t}{JJ_t}$$

The NK Model 3: Price and Inflation Dynamics

Details of algebra in Appendix 5 of the Notes

- **Price dynamics** as difference equations: Then we can express the foc above as **difference equations**:

$$
\begin{aligned}
\frac{P_t^0}{P_t} &= \frac{J_t}{JJ_t} \\
J_t - \xi E_t[\Lambda_{t,t+1}\Pi_{t+1}^{\zeta} J_{t+1}] &= \left(\frac{1}{1-\frac{1}{\zeta}}\right) Y_t MC_t MS_t \\
JJ_t - \xi E_t[\Lambda_{t,t+1}\Pi_{t+1}^{\zeta-1} JJ_{t+1}] &= Y_t
\end{aligned}
$$

- By the law of large numbers the evolution of the price index is given by the D-S Aggregator:

$$P_t^{1-\zeta} = \xi P_{t-1}^{1-\zeta} + (1-\xi)(P_t^O)^{1-\zeta}$$

- Defining $[t-1, t]$ inflation by $\Pi_t \equiv \frac{P_t}{P_{t-1}}$ it follows that

$$1 = \xi \Pi_t^{\zeta-1} + (1-\xi)\left(\frac{J_t}{JJ_t}\right)^{1-\zeta}$$

The NK Model 3: Price Dispersion

See Appendix 5 of Notes

- **Real marginal costs** are given by

$$MC_t = \frac{P_t^W}{P_t}$$

- In the RBC model $P_t^W = P_t$ so real marginal costs are fixed at unity.
- Note we have added a *mark-up shock* MS_t to real marginal costs.
- The next change is the introduction of **price dispersion** Δ_t which reduces output. As shown in Appendix 5 of the Notes we have:

$$\begin{aligned} Y_t &= \frac{(A_t H_t)^\alpha K_{t-1}^{1-\alpha}}{\Delta_t} \\ \Delta_t &\equiv \int_0^1 (P_t(j)/P_t)^{-\zeta} dj = \xi \Pi_t^\zeta \Delta_{t-1} + (1-\xi)\left(\frac{J_t}{JJ_t}\right)^{-\zeta} \end{aligned}$$

- If steady state inflation is non-zero, Δ_t is of second order so for a first-order approximation can be ignored.

The NK Model 3: Monetary Policy

- The nominal interest rate is given by the following Taylor-type rules

$$\textbf{Implementable} : \log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right)$$
$$+ \quad (1-\rho_r)\left[\theta_\pi \log\left(\frac{\Pi_t}{\Pi}\right) + \theta_y \log\left(\frac{Y_t}{Y}\right)\right] + \epsilon_{M,t} \quad \textbf{or}$$

$$\textbf{Conventional Taylor} : \log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right)$$
$$+ \quad (1-\rho_r)\left[\theta_\pi \log\left(\frac{\Pi_t}{\Pi}\right) + \theta_y \log\left(\frac{Y_t}{Y_t^F}\right)\right] + \epsilon_{M,t}$$

  where Y_t^F is the flexi-price level of output (as in RBC model) and $\epsilon_{M,t}$ is a monetary policy i.i.d shock.
- The 'implementable' form stabilizes output about its steady state, the conventional form about Y_t^F. Then θ_π and θ_y are the long-run elasticities of the inflation and output respectively with respect to the interest rate. The "Taylor principle" requires $\theta_\pi > 1$.

NK Model 3: Euler Equation and Supply of Capital

- With a nominal side of the model we need to distinguish between the *ex ante* nominal gross interest rate $R_{n,t}$ set at time t and the *ex post* real interest rate, R_t. These are related by the *Fischer equation*

$$R_t = \frac{R_{n,t-1}}{\Pi_t}$$

- The stochastic Euler equation must now take the form

$$U_{C,t} = \mathbb{E}_t \left[\frac{R_{n,t}}{\Pi_{t+1}} U_{C,t+1} \right] \Rightarrow 1 = E_t[R_{t+1}\Lambda_{t,t+1}]$$

- Household supply of capital is now given by

$$\begin{aligned} \mathbb{E}_t[\Lambda_{t,t+1}R_{t+1}] &= \frac{\mathbb{E}_t \left[\Lambda_{t,t+1}[(1-\alpha)\frac{P^W_{t+1}}{P_{t+1}}\frac{Y^W_{t+1}}{K_t} + (1-\delta)Q_{t+1}] \right]}{Q_t} \\ &= \mathbb{E}_t[\Lambda_{t,t+1}R_{K,t+1}] \end{aligned}$$

Exogenous Shock Processes

- In a zero-growth steady state there are three exogenous AR1 shock processes to technology, government spending and the mark-up shock, and an i.i.d monetary policy shock:

$$\begin{aligned}\log A_t - \log A &= \rho_A(\log A_{t-1} - \log A) + \epsilon_{A,t} \\ \log G_t - \log G_t &= \rho_G(\log G_{t-1} - \log G) + \epsilon_{G,t} \\ \log MS_t - \log MS &= \rho_{MS}(\log MS_{t-1} - \log MS) + \epsilon_{MS,t}\end{aligned}$$

- Other shocks are possible such as to investment and preferences and capital quality. (See models with financial frictions).

The NK Model 3 Steady State: Real Component

Consider first the case of a zero net inflation steady state ($\Pi = 1$). Then the *real* component of the steady state is that of the RBC model with investment adjustment costs and a wholesale-retail mark-up $\frac{P^W}{P}$:

$$\begin{aligned}
Q &= 1 \text{ (because } S(X) = S'(X) = S(0) = S'(0) = 0) \\
R &= \frac{1}{\beta} \\
Y^W &= (AH)^{\alpha} K^{1-\alpha} \\
\Delta &= 1 \\
Y &= \frac{Y^W}{\Delta} = Y^W \text{ since} \Delta = 1 \\
\frac{\varrho C}{(1-\varrho)(1-H)} &= W \\
\frac{\alpha P^W}{P} &= W \\
\frac{PK}{P^W Y^W} &= \frac{1-\alpha}{R-1+\delta}
\end{aligned}$$

NK Model 3 Steady State: Nominal Component

- For a non-zero steady state net inflation rate (gross rate $\Pi > 1$), for the *nominal* side we have

$$
\begin{aligned}
J(1-\beta\xi\Pi^{\zeta}) &= \frac{YMC}{\left(1-\frac{1}{\zeta}\right)} \\
JJ(1-\beta\xi\Pi^{\zeta-1}) &= Y \\
\frac{J}{JJ} &= \left(\frac{1-\xi\Pi^{\zeta-1}}{1-\xi}\right)^{\frac{1}{1-\zeta}} \\
MC &= \left(1-\frac{1}{\zeta}\right)\frac{J(1-\beta\xi\Pi^{\zeta})}{JJ(1-\beta\xi\Pi^{\zeta-1})} \\
\Delta &= \frac{(1-\xi)^{\frac{1}{1-\zeta}}(1-\xi\Pi^{\zeta-1})^{\frac{-\zeta}{1-\zeta}}}{1-\xi\Pi^{\zeta}}
\end{aligned}
$$

- For a zero-inflation steady state $\Pi = 1$ we arrive $\frac{J}{JJ} = \Delta = 1$ and $MC = \left(1-\frac{1}{\zeta}\right)$. Note there is a long-run inflation-output trade-off (see Ascari and Ropele (2007)).

NK Model 3 with Habit

- External habit introduces output persistence but is arguably a more plausible form of utility
- Introducing external habit ('keeping up with the Jones') for household j we now have

$$
\begin{aligned}
U_t^j &= \frac{(C_t^j - \chi C_{t-1})^{(1-\varrho)}(1-H_t)^{\varrho})^{1-\sigma_c} - 1}{1-\sigma_c} \\
U_{C,t}^j &= (1-\varrho)(C_t^j - \chi C_{t-1})^{(1-\varrho)(1-\sigma_c)-1}((1-H_t)^{\varrho(1-\sigma_c)}) \\
U_{L,t}^j &= -\varrho(C_t^j - \chi C_{t-1})^{(1-\varrho)(1-\sigma_c)}(1-H_t)^{\varrho(1-\sigma_c)-1}
\end{aligned}
$$

  where aggregate per capita consumption C_{t-1} is taken as given.
- Then in an equilibrium of identical households, $C_t^j = C_t$.

NK Model 3 with Indexation

- Price Indexation introduces inflation persistence in an ad hoc fashion
- With indexing by an amount γ the optimal price-setting first-order condition for a firm j setting a new optimized price $P_t^0(j)$ becomes

$$P_t^0(m) E_t \left[\sum_{k=0}^{\infty} \xi^k \frac{\Lambda_{t,t+k}}{P_{t+k}} Y_{t+k}(m) \left(\frac{P_{t+k-1}}{P_{t-1}} \right)^{\gamma} \right]$$

$$= \quad \frac{1}{(1 - 1/\zeta)} E_t \left[\sum_{k=0}^{\infty} \xi^k \Lambda_{t,t+k} MC_{t+k} Y_{t+k}(m) \right]$$

- Price dynamics are now given by

$$\frac{P_t^0}{P_t} = \frac{J_t}{JJ_t}$$

$$JJ_t - \xi\beta E_t[\Lambda_{t,t+1} \tilde{\Pi}_{t+1}^{\zeta-1} JJ_{t+1}] = Y_t \text{ where } \tilde{\Pi}_t \equiv \frac{\Pi_t}{\Pi_{t-1}^{\gamma}}$$

$$J_t - \xi\beta E_t[\Lambda_{t,t+1} \tilde{\Pi}_{t+1}^{\zeta} J_{t+1}] = \frac{1}{1 - \frac{1}{\zeta}} MC_t MS_t Y_t$$

The Workhorse NK Model 1

- The workhorse NK model is the above model with no capital, no habit and no indexing. It is summarized as follows.
- Instead of a non-separable utility function: $U_t = \frac{(C_t^{(1-\varrho)}(1-H_t)^{\varrho})^{1-\sigma_c}-1}{1-\sigma_c}$, it assumes a separable utility function:

$$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\psi}}{1+\psi}$$

  where ψ is the Frisch parameter.
- We log-linearize the model about the non-stochastic steady state *zero-growth, zero-inflation*.
- At the centre in the linearized NK Phillips Curve:

$$\pi_t = \beta E_t \pi_{t+1} + \lambda(mc_t + ms_t)$$

  where λ is a constant given by

$$\lambda = \frac{(1-\beta\xi)(1-\xi)}{\xi}$$

The Linearized Workhorse NK Model 1

The log-linearization of the whole model about the non-stochastic steady state *zero-growth, zero-inflation* is given by

$$
\begin{aligned}
\text{AR1 Shocks} &: \quad x_t = \rho_X x_{t-1} + \epsilon_{X,t}\,; \quad X = A, G, MS \\
\text{Euler Equation} &: \quad \mathbb{E}_t[u_{C,t+1}] = u_{C,t} - \mathbb{E}_t(r_{n,t} - \pi_{t+1}) \\
\text{Labour Supply} &: \quad w_t = -u_{H,t} - u_{C,t} \\
\text{Real Marginal Cost} &: \quad mc_t = p_t^w - p_t \\
\text{Labour Demand} &: \quad w_t = mc_t + y_t - h_t \\
\text{Production Function} &: \quad y_t = \alpha(a_t + h_t) \\
\text{Equilibrium} &: \quad y_t = c_y c_t + g_y g_t \\
\text{Philips Curve} &: \quad \pi_t = \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\beta\xi)(1-\xi)}{\xi}(mc_t + ms_t) \\
\text{Monetary Rule} &: \quad r_{n,t} = \rho_r r_{n,t-1} + (1-\rho_r)(\theta_\pi \pi_t + \theta_y y_t) + \epsilon_{M,t} \\
\text{Marginal utility of } C_t &: \quad u_{C,t} = -\sigma c_t \\
\text{Marginal utility of } H_t &: \quad u_{H,t} = -\psi h_t
\end{aligned}
$$

The Linear New Keynesian Phillips Curve

- Further insight into the NK Phillips curve can be obtained by solving it forward in time to obtain

$$\begin{aligned}\pi_t &= \beta E_t \underbrace{(\beta E_{t+1}\pi_{t+2} + \lambda(mc_{t+1} + ms_{t+1})}_{\pi_{t+1}} + \lambda(mc_t + ms_t) \\ &= \lambda(mc_t + ms_t + \beta E_t(mc_{t+1} + ms_{t+1}) + \beta^2 E_t \pi_{t+2}\end{aligned}$$

- This uses $E_t E_{t+1}\pi_{t+2} = E_t \pi_{t+2}$ (the "law of iterated expectations").
- Re-iterating we arrive at

$$\pi_t = \lambda E_t \left[\sum_{i=0}^{\infty} \beta^i (mc_{t+i} + ms_{t+i}) \right]$$

- Thus the inflation rate is *proportional to the discounted sum of the future real marginal costs with mark-up shocks* from producing the differentiated good.

The Flexi-Price Economy

- This is the limit as $\xi \to 0$ in which case $mc_t = mc_t^F = 0$ denoting the flexi-price model with a superscript F. It is given by

$$\begin{aligned} \text{Euler Equation}: \sigma_c \mathbb{E}_t[c_{t+1}^F] &= \sigma_c c_t^F - r_t^F &(3)\\ \text{Labour Supply}: \quad w_t = -u_{H,t} - u_{C,t} &= \psi h_t^F + \sigma_c c_t^F &(4)\\ \text{Labour Demand}: \quad w_t^F &= y_t^F - h_t^F &(5)\\ \text{Production Function}: \quad y_t^F &= \alpha(a_t + h_t^F) &(6)\\ \text{Equilibrium}: \quad y_t^F &= c_y c_t^F + g_y g_t &(7) \end{aligned}$$

with exogenous shocks a_t and g_t processes as before.

- Equations (4) – (7) define an equilibrium in n_t^F, w_t^F, y_t^F and c_t^F given a_t and g_t. Then (3) gives the *natural rate of interest* as:

$$r_t^F = \sigma_c(\mathbb{E}_t[c_{t+1}^F] - c_t^F) \tag{8}$$

- The flexi-price model which has a price mark-up and RBC models still differ in the steady state. But as the price elasticity $\zeta \to \infty$ the mark-up disappears and the flexi-price NK and RBC models are same.

The Three Equation Workhorse NK Model 1

- With $g_y = G_t = 0$ and 'Implementable' Taylor Nominal Interest Rate Rule the model is:

$$\pi_t \quad = \quad \beta E_t \pi_{t+1} + \kappa(y_t + ms_t)$$

- **"IS Curve": Aggregate Demand**

$$E_t y_{t+1} = y_t + \frac{1}{\sigma_c}(r_{n,t} - E_t \pi_{t+1})$$

- **Implementable Taylor Nominal Interest Rate Rule**

$$r_{n,t} = \rho_r r_{n,t-1} + (1 - \rho_r)(\theta_\pi \pi_t + \theta_y y_t)) + \epsilon_{M,t}$$

- where $y_t \equiv \log Y_t/Y$, $\pi_t = \log \Pi_t/\Pi$ etc and κ is given by

$$\kappa \equiv \frac{(1-\beta\xi)(1-\xi)}{\xi} \frac{\alpha(1+\psi)}{1+\psi+\alpha(\sigma_c - 1)} \tag{9}$$

- y_t^F and r_t^F are not required.

The Three Equation Workhorse NK Model 1

- Let $x_t \equiv y_t - y_t^F$ be the *output gap*
- With a Conventional Taylor Nominal Interest Rate Rule the model is:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(x_t + ms_t)$$

- **"IS Curve": Aggregate Demand**

$$E_t x_{t+1} = x_t + \frac{1}{\sigma_c}(r_{n,t} - E_t \pi_{t+1} - r_t^F)$$

- The natural rate of interest r_t^F and flexi-price output y_t^F are given by

$$r_t^F = \sigma_c \mathbb{E}_t y_{t+1}^F - y_t^F \quad (10)$$

$$y_t^F = \frac{\alpha(1+\psi)}{1+\psi+\alpha(\sigma_c - 1)} a_t \quad (11)$$

- **Conventional Taylor Nominal Interest Rate Rule**

$$r_{n,t} = \rho_r r_{n,t-1} + (1-\rho_r)(\theta_\pi \pi_t + \theta_x x_t)) + \epsilon_{M,t}$$

- See Notes Section 7.4 for algebraic details.

Understanding the Impulse Responses

- Linearized models can be used to understand the impulse response functions set out below.
- Consider a simpler version of **model 3** where $\sigma = 1$ (Cobb-Douglas utility) and $\chi = 0$ (no external habit).
- Then the linearized form (see Notes, Section 6.2.7) gives:

$$
\begin{aligned}
\textbf{labour supply} : w_t &= c_t + \frac{H}{1-H} h_t^s \\
\textbf{production function} : y_t &= a_t + \alpha h_t^d + (1-\alpha) k_{t-1} \\
\textbf{output equilibrium} : c_t &= \frac{1}{c_y}(y_t - i_y i_t - g_y g_t) \\
\textbf{real marginal cost} : mc_t &= w_t + h_t^d - y_t^w = w_t + h_t^d - y_t
\end{aligned}
$$

- Hence

$$w_t = y_t - h_t^d + mc_t = a_t + (1-\alpha)(k_{t-1} - h_t^d) + mc_t$$

Understanding the Impulse Responses

- The real marginal cost (the inverse of the retail mark-up), mc_t, is given by the Phillip's curve as

$$mc_t = \frac{\pi_t - \beta \mathbb{E}_t \pi_{t+1}}{\lambda}$$

which implies that a temporary **increase in inflation** with $\mathbb{E}_t \pi_{t+1} < \pi_t$ (a looser monetary policy) will see marginal costs **increase**.

- Similarly a temporary **decrease in inflation** will see marginal costs **decrease**.

- In the labour market equilibrium $h_t^s = h_t^d = h_t$. In the steady state at $t-1$, $k_{t-1} = 0$ and a little algebra sees **equilibrium hours** as

$$h_t = \frac{\left(1 - \frac{1}{c_y}\right) a_t + i_t + \left(\frac{g_y}{c_y}\right) g_t + mc_t}{\frac{\alpha}{c_y} + \frac{H}{1-H} + 1 - \alpha} \quad (12)$$

Understanding the Impulse Responses

- Thus hours *rise* on impact with a positive technology shock (a_t rises) keeping $g_t = 0$ iff

$$\underbrace{\left(1 - \frac{1}{c_y}\right)}_{\text{negative}} a_t + i_t + mc_t > 0 \tag{13}$$

  which since $c_y < 1$, requires as a necessary condition $i_t + mc_t > 0$.

- For the RBC model $mc_t = 0$ and from the irfs below i_t rises with an increase in a_t. Then provided that adjustment costs ϕ_X are absent or small, hours will then **increase**.
- With the NK model by contrast a positive technology shock leads to an immediate fall in inflation π_t (a tightening of monetary policy) and $mc_t < 0$, so the possibility of a **fall** in hours appears.

Understanding the Impulse Responses

- For the real wage rate w_t it is straightforward to show:

$$w_t = \underbrace{\left(1 - \frac{(1-\alpha)(1-\frac{1}{c_y})}{\frac{\alpha}{c_y} + \frac{H}{1-H} + 1 - \alpha}\right)}_{\text{positive}} (a_t + mc_t) - \frac{(1-\alpha)(i_t + \frac{g_y}{c_y} g_t)}{\frac{\alpha}{c_y} + \frac{H}{1-H} + 1 - \alpha} \tag{14}$$

- In a similar manner to hours, this equation can be used to explain the response of the real wage to impulses a_t and g_t in the irfs.
- In response to a positive technology shock the real wage will *fall* in the NK model iff $a_t + mc_t < 0$ i.e., the effect of a fall in inflation policy where $mc_t < 0$ outweighs the productivity effect.
- For the RBC model $mc_t = 0$ and the first term is positive whereas the term in i_t is negative.
- For high α (labour share), the former outweighs the latter - see figure.

Understanding the Impulse Responses

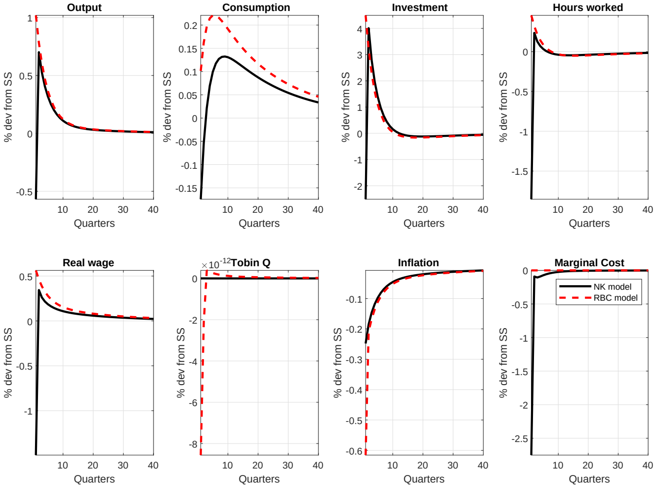

Figure: Technology Shock: Zero Investment Adjustment Costs ($\phi_X = 0$)

Understanding the Impulse Responses

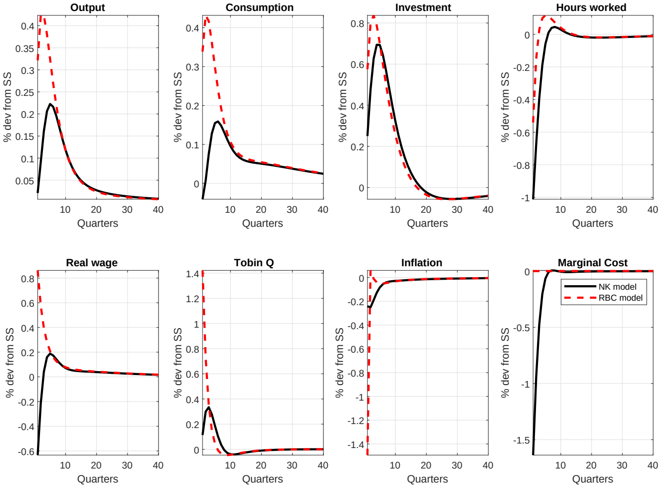

Figure: Technology Shock: High Investment Adjustment Costs ($\phi_X = 4$)

The Full SW Model 2

- We now introduce six new features
  - A different form of preferences
  - Wage stickiness
  - Capacity Utilization
  - Fixed costs of converting homogeneous output into differentiated goods.
  - A non-zero balanced growth steady state with a stochastic trend.
  - Internal rather than External Habit
  - Unemployment (or rather underemployment) in a simple way
- Smets and Wouters (2007) has one more feature: *Kimball aggregators* as in Kimball (1995) and Klenow and Willis (2016) generalize the Dixit-Stigliz aggregator. But we do not include this feature.
- Here we concentrate on SW preferences, internal vs external habit and wage stickiness - full details of remaining features (including stationarization) are in the Notes

SW Preferences

- As with CD preferences, those of SW are compatible with balanced growth (see King *et al.* (1988)).
- In stationarized form, with external habit in consumption, household j has a single-period *non-separable* utility

$$U_t^j = \frac{(C_t^j - \chi C_{t-1}/(1+g_t))^{1-\sigma_c} \exp\left(\frac{(\sigma_c-1)(H_t^j)^{1+\psi}}{1+\psi}\right) - 1}{1-\sigma_c}\,; \quad \chi \in [0,1)$$

$$\to \log(C_t^j - \chi C_{t-1}/(1+g_t)) + \frac{(H_t^j)^{1+\psi}}{1+\psi} \text{ as } \sigma_c \to 1$$

where C_{t-1} is aggregate per capita consumption

- Whereas with internal habit we have

$$U_t^j = \frac{(C_t^j - \chi C_{t-1}^j/(1+g_t))^{1-\sigma_c} \exp\left(\frac{(\sigma_c-1)(H_t^j)^{1+\psi}}{1+\psi}\right) - 1}{1-\sigma_c}\,; \quad \chi \in [0,1)$$

$$\to \log(C_t^j - \chi C_{t-1}^j/(1+g_t)) + \frac{(H_t^j)^{1+\psi}}{1+\psi} \text{ as } \sigma_c \to 1$$

SW Preferences

- Defining an instantaneous marginal utility by

$$U_{C,t} = (C_t - \chi C_{t-1}/(1+g_t))^{-\sigma_c} \exp\left(\frac{(\sigma_c - 1)H_t^{1+\psi}}{1+\psi}\right)$$

- Then in a symmetric equilibrium, household first-order conditions are

$$1 = \mathbb{E}_t\left[R_{t+1}\Lambda_{t,t+1}\right] \text{ where } \Lambda_{t,t+1} \equiv \beta_{g,t+1}\frac{\lambda_{t+1}}{\lambda_t}$$

$$U_{H,t} = -H_t^{\psi}(C_t - \chi C_{t-1}/(1+g_t))^{1-\sigma_c} \exp\left(\frac{(\sigma_c - 1)H_t^{1+\psi}}{1+\psi}\right)$$

$$\frac{U_{H,t}}{\lambda_t} = -W_t$$

where for external habit and internal habit respectively we have

$$\lambda_t = U_{C,t}$$

$$\lambda_t = U_{C,t} - \beta\chi\mathbb{E}_t[U_{C,t+1}]$$

SW Preferences

- Note that unless $\sigma_c \to 1$, the *separable* utility function:

$$U_t^j = \frac{(C_t^j - \chi C_{t-1}/(1+g_t))^{1-\sigma_c} - 1}{1-\sigma_c} + \frac{(H_t^j)^{1+\psi}}{1+\psi}$$

  in Woodford (2003) is *not* compatible with balanced growth.
- Parameter ψ is referred to by Smets and Wouters (2007) as the labour supply elasticity.

Wage Stickiness I

- To introduce wage stickiness we now assume that each household supplies homogeneous labour at a nominal wage rate $W_{h,t}$ to a monopolistic trade-union
- She then differentiates the labour and sells type $H_t(j)$ at a nominal wage $W_{n,t}(j) > W_{h,t}$ to a labour packer in a sequence of Calvo staggered nominal wage contracts.
- The real wage is then defined as $W_t \equiv \frac{W_{n,t}}{P_t}$. We now have to distinguish between *price inflation* which now uses the notation $\Pi_t^p \equiv \frac{P_t}{P_{t-1}}$ and *wage inflation*. $\Pi_t^w \equiv \frac{W_{n,t}}{W_{n,t-1}}$
- As with price contracts we employ Dixit-Stiglitz quantity and price aggregators. Calvo probabilities are now ξ_p and ξ_w for price and wage contracts respectively. Similarly J_t, JJ_t are replaced with J_t^p, JJ_t^p and J_t^w, JJ_t^w to model price and nominal wage dynamics respectively with dispersions Δ_t^p and Δ_t^w.

Wage Stickiness II

- The competitive labour packer forms a composite labour service according to $H_t = \left(\int_0^1 H_t(j)^{(\mu-1)/\mu} dj\right)^{\mu/(\mu-1)}$ and sells onto the intermediate firm. where μ is the elasticity of substitution. For each j, the labour packer chooses $H_t(j)$ at a wage $W_{n,t}(j)$ to maximize H_t given total expenditure $\int_0^1 W_{n,t}(j)H_t(j)dj$.
- This results in a set of labour demand equations for each differentiated labour type j with wage $W_{n,t}(j)$ of the form

$$H_t(j) = \left(\frac{W_{n,t}(j)}{W_{n,t}}\right)^{-\mu} H_t^d \tag{15}$$

where $W_{n,t} = \left[\int_0^1 W_{n,t}(j)^{1-\mu} dj\right]^{\frac{1}{1-\mu}}$ is the aggregate wage index.

- H_t and $W_{n,t}$ are Dixit-Stigliz aggregators for the labour market corresponding to Y_t and P_t for the output market in Model 3.

Wage Setting by the Trade-Union I

- Wage setting by the trade-union again follows the standard Calvo framework supplemented with indexation.
- At each period there is a probability $1 - \xi_w$ that the wage is set optimally. The optimal wage derives from maximizing discounted profits.
- For those trade-unions unable to reset, wages are indexed to last period's aggregate inflation, with wage indexation parameter γ_w.
- Then as for price contracts the wage rate trajectory with no re-optimization is given by $W^O_{n,t}(j)$, $W^O_{n,t}(j)\left(\frac{P_t}{P_{t-1}}\right)^{\gamma_w}$, $W^O_{n,t}(j)\left(\frac{P_{t+1}}{P_{t-1}}\right)^{\gamma_w}, \cdots$.
- The trade union then buys homogeneous labour at a nominal price $W_{h,t}$ and converts it into a differentiated labour service of type j.

Wage Setting by the Trade-Union II

- The trade union time t then chooses $W^O_{n,t}(j)$ to maximize real profits

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi_w^k \frac{\Lambda_{t,t+k}}{P_{t+k}} H_{t+k}(j) \left[W^O_{n,t}(j) \left(\frac{P_{t+k-1}}{P_{t-1}} \right)^{\gamma_w} - W_{h,t+k} \right]$$

where using (15) with indexing $H_{t+k}(j)$ is given by

$$H_{t+k}(j) = \left(\frac{W^O_{n,t}(j)}{W_{n,t+k}} \left(\frac{P_{t+k-1}}{P_{t-1}} \right)^{\gamma_w} \right)^{-\mu} H^d_{t+k}$$

- By analogy with price-setting this leads to the optimal real wage

$$\frac{W^O_{n,t}}{P_t} = \frac{1}{(1-1/\mu)} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \xi_w^k \Lambda_{t,t+k} \left(\Pi^w_{t,t+k} \right)^{\zeta} H^d_{t+k} \frac{W_{h,t+k}}{P_{t+k}}}{\mathbb{E}_t \sum_{k=0}^{\infty} \xi_w^k \Lambda_{t,t+k} \left(\Pi^w_{t,t+k} \right)^{\zeta} \left(\Pi^p_{t,t+k} \right)^{-1} H^d_{t+k}}$$

- Then by the law of large numbers:

$$W^{1-\mu}_{n,t} = \xi_w \left(W_{n,t-1} \left(\frac{P_t}{P_{t-1}} \right)^{\gamma_w} \right)^{1-\mu} + (1-\xi_w)(W^0_{n,t}(j))^{1-\mu}$$

Wage Dynamics

- Now define

$$\Pi_t^p \equiv \frac{P_t}{P_{t-1}};\ \Pi_t^w \equiv \frac{W_t}{W_{t-1}}\ \tilde{\Pi}_t^w \equiv \frac{\Pi_t^w}{(\Pi_{t-1}^p)^{\gamma_w}};\ \tilde{\Pi}_t^p(\gamma) \equiv \frac{\Pi_t^p}{(\Pi_{t-1}^p)^{\gamma}}$$

- Then as for price dynamics we have

$$\begin{aligned}
\frac{W_t^O}{W_{n,t}} &= \frac{J_t^w}{W_t JJ_t^w} \\
JJ_t^w - \xi_w \mathbb{E}_t \left[\Lambda_{t,t+1} \frac{(\tilde{\Pi}_{t,t+1}^w)^{\mu}}{\tilde{\Pi}_{t,t+1}^p(\gamma_w)} JJ_{t+1}^w \right] &= H_t^d \\
J_t^w - \xi_w \mathbb{E}_t \left[\Lambda_{t,t+1} \tilde{\Pi}_{w,t+1}^{\mu} J_{t+1}^w \right] &= -\frac{\mu}{\mu - 1} MRS_t MS_{w,t} H_t^d
\end{aligned}$$

where $MRS_t = -\frac{U_{H,t}}{U_{C,t}} = \frac{W_{h,t}}{P_t}$.

Dispersion Dynamics

- The set-up is completed with a relationship between real wage growth, price inflation Π_t^p and wage inflation Π_t^w

$$W_t = \frac{\Pi_t^w}{\Pi_t^p} W_{t-1};$$

- wage dispersion Δ_t^w (with price dispersion now Δ_t^p):

$$\Delta_t^w = \xi_w (\tilde{\Pi}_t^w)^\mu \Delta_{t-1}^w + (1-\xi_w)\left(\frac{W_t^O}{W_{n,t}}\right)^{-\mu}$$

- and a final composite output - retail output relationship:

$$Y_t = \frac{Y_t^W - F}{\Delta_t^p}$$
$$Y_t^W = (A_t H_t^d)^\alpha (u_t K_{t-1})^{1-\alpha}$$
$$H_t^d = \frac{H_t}{\Delta_t^w}$$

- Note that *underemployment* is $H_t - H_t^d$

Dynare Model Files

In **folder NK**

- **NKlinear.mod**: Linear NK model 3 with habit and indexing options. With a flexi-price option and therefore set up for both an implementable rule and conventional Taylor rule.
- **NK.mod**: Non-linear NK model 3 with habit, indexing options and price dispersion. With a flexi-price option and therefore set up for both an implementable rule and conventional Taylor rule.
- **SW_NK.mod**: Non-linear SW NK model 2. This uses an **external steady state** - see Day 1. Set-up has external or internal habit option; Indexing takes place in both the steady state and out of the steady state (unlike Smets and Wouters (2007). No flexi-price option and therefore only set up for an implementable rule.
- **graphs_irfs_compare_NK** Graph plotter for irfs of non-linear or linear NK model

Dynare Modelling Exercises I

1. Examine the role of habit and indexing in the **NKlinear.mod** model by comparing impulse responses for zero, medium and high parameter settings of χ and γ.
2. Compare the impulse response functions and the variance decomposition of RBC with external habit and NK models, both with investment costs, for low, medium and high values of the investment cost parameter $\phi_X = 0.1, 2.0, 4.0$.[1] Use the slides on understanding irfs to interpret the results.
3. The easiest way to do this is to first run the NK model with perfect competition in the retail sector ($\zeta = 10000$, a large number) and $\xi = 0$. This is in effect the "RBC model". Then reset ζ and ξ as before for the NK model. In fact the flexi-sticky price option does this for you. Does the monetary shock have real effects in the RBC case?
4. Compare the impulse response functions of NKlinear with implementable and conventional Taylor rules.

[1] The responses of hours to a technology shock relates to the hours-technology debate (see Galí and Rabanal (2005), Whelan (2009) and Cantore *et al.* (2010))

Dynare Modelling Exercises II

1. Modify the code **NK.mod** by introducing a preference shock (PS_t) in utility, and an investment specific technology shock (IS_t). Modified equilibrium conditions are:

$$
\begin{aligned}
U_t &= PS_t \frac{((C_t - \chi C_{t-1})^{(1-\varrho)}(1-H_t)^{\varrho})^{1-\sigma_c} - 1}{1-\sigma_c} \\
U_{C,t} &= PS_t(1-\varrho)(C_t - \chi C_{t-1})^{(1-\varrho)(1-\sigma_c)-1}((1-H_t)^{\varrho(1-\sigma_c)} \\
U_{H,t} &= -PS_t \varrho (C_t - \chi C_{t-1})^{(1-\varrho)(1-\sigma_c)}(1-H_t)^{\varrho(1-\sigma_c)-1} \\
K_t &= (1-\delta)K_{t-1} + (1-S(X_t))I_t \, IS_t \\
1 &= Q_t \, IS_t(1 - S(X_t) - X_t S'(X_t)) \\
&+ E_t\left[\Lambda_{t,t+1} \, Q_{t+1} S'(X_{t+1}) X_{t+1}^2 \, IS_{t+1}\right]
\end{aligned}
$$

2. Discuss the irfs for these extra shocks.

Dynare Modelling Exercises III

In **NK_SW.mod** compare impulse responses of the model with the following parameter choices

1. $\xi_p = \xi_w = 0.7$ (sticky prices and wages)
2. $\xi_p = 0.7$, $\xi_w = 0$ (sticky prices, flexi wages as in Model 3)
3. $\xi_p = 0$, $\xi_w = 0.7$ (flexi wages, sticky prices)

Do the same for the sticky prices and wages model comparing external and internal habit using the pre-processor option.

What do you conclude from these two exercises?

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