The Science and Art of DSGE Modelling
A Foundations Course

Linearization

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Linearization: First Order Taylor Series Expansion

- Stability analysis in dynamic models is based on a linear (first-order) approximation about some baseline.
- The easiest way to set up a model in dynare is in linearized form about a steady state.
- The technique of linearization uses a Taylor series expansion.
- Consider a general function of two variables $F(X_t, Y_t)$.
- Then up to first-order terms we have

$$F(X_t, Y_t) \approx F(X, Y) + \frac{\partial F}{\partial X_t}(X_t - X) + \frac{\partial F}{\partial Y_t}(Y_t - Y) \tag{1}$$

where partial derivatives are evaluated at the steady-state values X, Y.

Linearization: Some Notation

- We now use the following notation

$$\frac{X_t - X}{X} = x_t \tag{2}$$

- We deal with *proportional deviations* unlike the dynare solution procedure above which used *absolute deviations* $\hat{\mathrm{Y}}_t \equiv \mathrm{Y}_t - \mathrm{Y}$. Unless Y_t is in logs, these are quite different.
- Then (1) becomes

$$f_t \approx \frac{X}{F}\frac{\partial F}{\partial X_t}x_t + \frac{Y}{F}\frac{\partial F}{\partial Y_t}y_t \tag{3}$$

- Note that

$$\log\left[\frac{X_t}{X}\right] = \log\left[1 + \frac{X_t - X}{X}\right] \approx x_t \tag{4}$$

which is why this technique is referred to as 'log-linearization'.

Absolute and Proportional Deviations in Dynare

- In our dynare codes we have defined variables such as $XX_t \equiv \frac{X_t}{X}$ written in dynare as X/STEADY_STATE(X).
- Then since dynare uses *absolute deviations* $\hat{XX}_t \equiv XX_t - XX$ and $XX = 1$, it follows that
- $\hat{XX}_t = XX_t - 1 = \frac{X_t - X}{X} \equiv x_t$ are now *proportional deviations* in X_t.
- Impulse reaction functions can now be interpreted as *percentage elasticities*.
- Variable XX_t in a non-linear set-up corresponds to X_t in the linearized one.

Linearization: Application

- Define lower case variables $x_t \approx \log \frac{X_t}{\bar{X}_t}$ if X_t has a long-run trend
- Or $x_t \approx \log \frac{X_t}{X}$ otherwise where X is the steady state value of a non-trended variable.
- Appendix 2 in the notes and the following slides illustrate this method for the Euler equation in both the RBC and NK models with CD utility.

Example of Linearization: The Euler Equation

- The non-linear without habit form is given by

$$U_{C,t} = \beta R_t \mathbb{E}_t \left[U_{C,t+1} \right] \quad (5)$$

$$U_{C,t} = (1-\varrho) C_t^{(1-\varrho)(1-\sigma_c)-1} (1-H_t)^{\varrho(1-\sigma_c)} \quad (6)$$

- Let's now apply the Taylor-Series approximation to (5). Applying

$$f(R_t, U_{C,t+1}) \approx f(R, U_C) + \frac{\partial f}{\partial R_t}(R_t - R) + \frac{\partial f}{\partial U_{C,t+1}}(U_{C,t+1} - U_C)$$

with the partial derivatives evaluated at the steady state, to the right-hand-side of (5), and using the steady-state $\beta R = 1$:

$$\begin{aligned} U_{C,t} &\approx U_C + \beta U_C (R_t - R) + \beta R \mathbb{E}_t [U_{C,t+1} - U_C] \\ &\approx U_C + U_C \frac{(R_t - R)}{R} + \mathbb{E}_t [U_{C,t+1} - U_C] \end{aligned}$$

- Hence

$$u_{C,t} \equiv \frac{U_{C,t} - U_C}{U_C} \approx r_t + \mathbb{E}_t [u_{C,t+1}]$$

Example of Linearization: The Euler Equation (cont)

- To linearize (6) it is quickest to first take logs to obtain

$$\log U_{C,t} = ((1-\varrho)(1-\sigma_c)-1)\log C_t + \varrho(1-\sigma_c)\log(1-H_t) + \text{constant}$$

Then subtracting the steady state from both sides of the equation:

$$\begin{aligned} \log U_{C,t} - \log U_C &= ((1-\varrho)(1-\sigma_c)-1)(\log C_t - \log C) \\ &+ \varrho(1-\sigma_c)(\log(1-H_t) - \log(1-H)) \end{aligned}$$

Now again use the Taylor series expansion to approximate

$$\log(1-H_t) - \log(1-H) \approx -\frac{1}{1-H}(H_t - H) = -\frac{H}{1-H}h_t$$

Hence we arrive at the linearization of section 4.2 in the notes:

$$u_{C,t} \approx -((\sigma_c - 1)(1-\varrho) + 1)c_t + (\sigma_c - 1)\varrho\frac{H}{1-H}h_t \qquad (7)$$

The Linearized RBC Model with External Habit

Given investment i_t and marginal utilities $u_{L,t}$ and $u_{C,t}$

$$\begin{aligned}
\text{Technology shock} &: \quad a_t = \rho_A a_{t-1} + \varepsilon_{A,t} \\
\text{Gov Spending shock} &: \quad g_t = \rho_G g_{t-1} + \varepsilon_{G,t} \\
\text{Capital Accumulation} &: \quad k_t = (1-\delta)k_{t-1} + \delta i_t \\
\text{Euler Equation} &: \quad \mathbb{E}_t[u_{C,t+1}] = u_{C,t} - r_t \\
\text{Labour Supply} &: \quad w_t = u_{L,t} - u_{C,t} \\
\text{Labour Demand} &: \quad w_t = y_t - h_t \\
\text{Production Function} &: \quad y_t = \alpha(a_t + h_t) + (1-\alpha)k_{t-1} \\
\text{Equilibrium} &: \quad y_t = c_y c_t + i_y i_t + g_y g_t
\end{aligned}$$

where recall that $x_t = \log \frac{X_t}{X} \approx \frac{X_t - X}{X}$ for any x_t.

The Linearized RBC Model - Continued

Marginal utilities of consumption and leisure and investment (with costs of adjustment) are given respectively by

$$
\begin{aligned}
u_{C,t} &= -(1+(\sigma_c-1)(1-\varrho))\underbrace{\left(\frac{c_t-\chi c_{t-1}}{1-\chi}\right)}_{\text{habit added}}+(\sigma_c-1)\varrho\frac{H}{1-H}h_t \\
u_{L,t} &= u_{C,t}+\underbrace{\left(\frac{c_t-\chi c_{t-1}}{1-\chi}\right)}_{\text{habit added}}+\frac{H}{1-H}h_t \\
r_t &= \frac{(R-1+\delta)(E_t y_{t+1}-k_t)+(1-\delta)E_t q_{t+1}}{R}-q_t \\
\left(1+\frac{1}{R}\right)i_t &= \frac{1}{R}E_t i_{t+1}+i_{t-1}+\frac{1}{S''(1)}q_t
\end{aligned}
$$

A government balanced budget constraint completes the model (but in the absence of tax distortions plays no role):

$$g_t = t_t$$

An Example of why Linearization is Useful

- Full paper and pen stability analysis possible for linear models and optimal policy for linear-quadratic problems (see Woodford (2003) and Galí (2015)).
- Consider the **Euler Equation**. Useful analysis can be done in the case of a logarithmic utility function ($\sigma_c = 1$). In this case $u_{C,t} \approx c_t$ and the the linear Euler equation becomes:

$$\mathbb{E}_t[u_{C,t+1}] = u_{C,t} - r_t \quad \Rightarrow c_t = -r_t + \mathbb{E}_t[c_{t+1}]$$

- Solving this forward in time gives

$$c_t = -r_t - \mathbb{E}_t\left[-r_{t+1} + \mathbb{E}_{t+1}[c_{t+2}]\right] = -r_t - \mathbb{E}_t[r_{t+1}] + \mathbb{E}_t\left[\mathbb{E}_{t+1}[c_{t+2}]\right]$$

- Then using $\mathbb{E}_t\left[\mathbb{E}_{t+1}[c_{t+2}]\right] = \mathbb{E}_t\left[c_{t+2}\right]$ (the "law of iterated expectations") and reiterating we arrive at:

$$c_t = -r_t - \mathbb{E}_t[r_{t+1}] - \mathbb{E}_t[r_{t+2}] - \cdots = -\sum_{i=0}^{\infty} \mathbb{E}_t[r_{t+i}]$$

Another Example of Why Linearization is Useful

- Gather the linearization of the labour supply choice and the marginal utility of leisure together:

$$
\begin{aligned}
w_t &= u_{L,t} - u_{C,t} \\
u_{L,t} &= u_{C,t} + \left(\frac{c_t - \chi c_{t-1}}{1 - \chi} \right) + \frac{H}{1 - H} h_t
\end{aligned}
$$

- It follow that $w_t = \left(\frac{c_t - \chi c_{t-1}}{1-\chi} \right) + \frac{H}{1-H} h_t$
- We now have an expression for the inverse elasticity of hours with respect to the wage keeping consumption fixed at its steady state.
- Then $c_t = c_{t-1}$ and the constant consumption inverse elasticity (the **Frisch elasticity**) is then $\frac{H}{1-H} = 1/2$ with $H = 1/3$.
- From empirical studies this is on the low side (estimates suggest an elasticity > 1).
- But JR rather than CD preferences used for the RBC model will enable us to target the Frisch elasticity

A Final Example: Understanding Impulse Responses

- The linearized model can be used to understand the impulse responses.
- Consider the simpler version where $\sigma = 1$ (Cobb-Douglas utility) and $\chi = 0$ (no external habit).
- Follow the reasoning in the Notes on Understanding Impulse Responses for NK model, for the flexi-price and wage limit (the RBC model)

$$\left(1 - \frac{1}{c_y}\right) a_t + i_t > 0 \tag{8}$$

- Thus since $c_y < 1$, this requires as a necessary condition $i_t > 0$.
- Return to this in the NK model of Day 2.
- Then provided that adjustment costs ϕ_X are absent or small, hours will increase.

Galí, J. (2015). *Monetary Policy, Inflation and the Business Cycle*. Princeton University Press, second edition.

Woodford, M. (2003). *Interest and Prices. Foundations of a Theory of Monetary Policy*. Princeton University Press.