The Science and Art of DSGE Modelling
A Foundations Course

The RBC Model

Kirill Shakhnov

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The RBC Model: General Points

- **Households** make an inter-temporal utility-maximizing choice of consumption and labour supply over time subject to a budget constraint. Net assets consists of capital owned by households and rented to firms
- **Firms** produce output according to a production technology and choose labour and capital inputs to minimize cost
- All firms and households are identical – the **representative agent** model. A huge literature now relaxes this assumption.
- Labour, output and financial **markets clear**
- We will allow for investment adjustment costs
- Can have exogenous non-zero balanced growth steady state path (bgp), but throughout we assume a **zero bgp**.

Household Utility

- Households choose between hours worked (H_t) and leisure (L_t) where L_t be the proportion of the total time available for work (say 16 hours per day) that consists of leisure time and H_t the proportion of this time spent at work. Then $L_t + H_t = 1$
- The **single-period utility** is

$$U = U(C_t, L_t)$$

where C_t is consumption and all variables are expressed in real terms relative to the price of retail output. We assume that

$$U_C > 0,\ \ U_L > 0\ \ U_{CC} \leq 0,\ \ U_{LL} \leq 0 \tag{1}$$

- In a stochastic environment, the **value function** of the representative household at time t is given by

$$V_t = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s U(C_{t+s}, L_{t+s}) \right] \ ;\ \ \beta \in (0,1) \tag{2}$$

The Household Optimization Problem

- For the household's problem at time t is to choose paths for consumption $\{C_t\}$, leisure, $\{L_t\}$, labour supply $\{H_t = 1 - L_t\}$, capital stock $\{K_t\}$ and investment $\{I_t\}$ to maximize V_t given by (2) given its budget constraint in period t with all variables in real terms:

$$B_t = R_{t-1}B_{t-1} + r_t^K K_{t-1} + W_t H_t - C_t - I_t - T_t \tag{3}$$

  where B_t is the *value* of the stock of one-period bonds (price$\times$number of bonds) at the end of period t.
- r_t^K is the rental rate for capital, W_t is the wage rate and R_{t-1} is the interest rate set in period $t-1$ paid in period t on bonds held at the end of period $t-1$.
- Note B_t and K_t are *end-of-period* bond and capital stock respectively.

The Household Optimization Problem (cont)

- Capital stock accumulates according to

$$K_t = (1-\delta)K_{t-1} + (1 - S(X_t))I_t\,; \tag{4}$$

$$X_t \equiv \frac{I_t}{I_{t-1}}\,;\;\; S',\, S'' \geq 0\,;\;\; S(1) = S'(1) = 0 \tag{5}$$

- In (5), $S(X_t)$ are investment adjustment costs.

- Then I_t units of output converts to $(1 - S(X_t))I_t$ of new capital sold at a real price Q_t ("Tobin's Q")

Solution to the Household Optimization Problem

First order conditions (See Appendix 1) are

$$\begin{aligned}
\text{Euler Consumption} &: \quad U_{C,t} = \beta R_t \mathbb{E}_t \left[U_{C,t+1} \right] \\
\text{Labour Supply} &: \quad \frac{U_{H,t}}{U_{C,t}} = -\frac{U_{L,t}}{U_{C,t}} = -W_t \\
\text{Leisure and Hours} &: \quad L_t \equiv 1 - H_t \\
\text{Investment FOC} &: \quad Q_t(1 - S(X_t) - X_t S'(X_t)) \\
&+ \quad \mathbb{E}_t \left[\Lambda_{t,t+1} \, Q_{t+1} S'(X_{t+1}) X_{t+1}^2 \right] = 1 \\
\text{Capital Supply} &: \quad \mathbb{E}_t \left[\Lambda_{t,t+1} R_{t+1}^K \right] = 1
\end{aligned}$$

where the gross return on capital is given by

$$R_t^K = \frac{\left[r_t^K + (1-\delta) Q_t \right]}{Q_{t-1}}$$

and $\Lambda_{t,t+1} \equiv \beta \frac{U_{C,t+1}}{U_{C,t}}$ is the *real stochastic discount factor* $[t, t+1]$.

The Euler Equation, the Stochastic Discount Factor

- In the Euler consumption equation, $U_{C,t} \equiv \frac{\partial U_t}{\partial C_t}$ denotes the marginal utility of consumption and $\mathbb{E}_t[\cdot]$ rational expectations assuming agents observe all current macroeconomic variables (i.e., 'perfect information').
- The Euler consumption equation equates the marginal utility from consuming one unit of income in period t with the discounted marginal utility in period $t+1$ from consuming the gross income acquired, R_t, by saving the income.
- It is convenient to write it as

$$1 = R_t \mathbb{E}_t \left[\Lambda_{t,t+1}\right] \tag{6}$$

where we recall $\Lambda_{t,t+1} \equiv \beta \frac{U_{C,t+1}}{U_{C,t}}$ is the real stochastic discount factor.
- Then we have the arbitrage condition

$$1 = R_t \mathbb{E}_t \left[\Lambda_{t,t+1}\right] = \mathbb{E}_t \left[\Lambda_{t,t+1} R^K_{t+1}\right] \tag{7}$$

Firms

- Output and the firm's behaviour is summarized by:

$$\begin{aligned} \text{Output} &: \quad Y_t = F(A_t, H_t, K_{t-1}) && (8) \\ \text{Labour Demand} &: \quad F_{H,t} = W_t && (9) \\ \text{Capital Demand} &: \quad F_{K,t} = r_t^K && (10) \end{aligned}$$

- (8) is a production function. Note again that K_t is *end-of-period t* capital stock.
- Equation (9), where $F_{H,t} \equiv \frac{\partial F_t}{\partial H_t}$, equates the marginal product of labour with the real wage.
- (10), where $F_{K,t} \equiv \frac{\partial F_t}{\partial K_t}$, equates the marginal product of capital with the rental rate r_t^K.

Output Equilibrium and Government Budget Constraint

- The model is completed with an output equilibrium and a balanced budget constraint with lump-sum taxes (T_t).

$$
\begin{aligned}
Y_t &= C_t + G_t + I_t \\
G_t &= T_t
\end{aligned}
$$

  where G_t is government spending.
- We now specify **functional forms** for production and utility and AR(1) processes for exogenous variables A_t and G_t. For production we assume a Cobb-Douglas function. The utility function is non-separable and consistent with a bgp when the inter-temporal elasticity of substitution, $1/\sigma_c$ is not unitary. It also satisfies conditions (1).

Functional Forms and Exogenous Processes

$$
\begin{aligned}
\text{Cobb-Douglas PF} &: \quad F(A_t, H_t, K_t) = (A_t H_t)^{\alpha} K_{t-1}^{1-\alpha} \\
\text{MPH} &: \quad F_H(A_t, H_t, K_t) = \frac{\alpha Y_t}{H_t} \\
\text{MPK} &: \quad F_K(A_t, H_t, K_t) = \frac{(1-\alpha) Y_t}{K_{t-1}} \\
\text{Technology} &: \quad \ln A_t - \ln \bar{A}_t = \rho_A (\ln A_{t-1} - \ln \bar{A}_{t-1}) + \epsilon_{A,t} \\
\text{Government} &: \quad \ln G_t - \ln \bar{G}_t = \rho_G (\ln G_{t-1} - \ln \bar{G}_{t-1}) + \epsilon_{G,t} \\
\text{Non-separable CRRA} &: \quad U_t = \frac{(C_t^{(1-\varrho)} (1-H_t)^{\varrho})^{1-\sigma_c} - 1}{1-\sigma_c} \\
&\rightarrow \quad (1-\varrho) \log C_t + \varrho \log(1-H_t) \text{ as } \sigma_c \rightarrow 1 \\
\text{MUC} &: \quad U_{C,t} = (1-\varrho) C_t^{(1-\varrho)(1-\sigma_c)-1} ((1-H_t)^{\varrho(1-\sigma_c)}) \\
\text{MUH} &: \quad U_{H,t} = -\varrho C_t^{(1-\varrho)(1-\sigma_c)} (1-H_t)^{\varrho(1-\sigma_c)-1} \\
\text{Inv Adj Costs} &: \quad S(X_t) = \phi_X (X_t - 1)^2
\end{aligned}
$$

Points to Note

- We now have an **equilibrium** in n independent dynamic equations solving for n macroeconomic variable
- See the Notes for a listing of equations the model above and for a simpler RBC model without investment adjustment costs
- This is the Dynare model code in the .mod files
- Dynare solves using a **perturbation method** based on a Taylor series expansion about a deterministic steady state
- Solution can be first, second (default) or third order
- The steady state can be **trended** - See Section 5.12 of the NK model Chapter in the Notes

The Deterministic Steady State in Dynare

- Now assume a non-trended **zero-growth** steady state, $\bar{A}_t = \bar{A}_{t-1} = A$, say and $\bar{G}_t = \bar{G}_{t-1} = G$. $K_t = K_{t-1} = K$, etc
- Computing the steady state is the most difficult part of the set-up!!
- Four approaches:
  - Use Initial Guesses - see Dynare Guide, but not recommended in general!
  - Solve for n variables in n unknowns using a Matlab equation solver fsolve . For the model above $n = 8$! (see Section 4.2 of Notes).
  - Exploit the recursive structure to solve for $m << n$ variables in Dynare
  - Obtain steady state completely recursively without need for a solution. **In fact we can do this for the RBC and NK models**.
- As we shall see in later models a completely recursive solution is not always possible.
- For more on calibration and setting up the steady state see Day 4.

Zero Growth Steady State in Recursive Form

The zero-growth steady state of the basic RBC model *with or without without investment costs* in recursive form is given by:

$$
\begin{aligned}
Q &= 1 \\
S &= 0 \\
X &= 1 \\
R &= \frac{1}{\beta} \\
\frac{K}{Y} &= \frac{(1-\alpha)}{R-1+\delta} \\
\frac{I}{Y} &= \frac{\delta K}{Y} = \frac{(1-\alpha)\delta}{R-1+\delta} = \frac{(1-\alpha)\delta}{R-1+\delta} \\
\frac{C}{Y} &= 1 - \frac{I}{Y} - \frac{G}{Y} = 1 - \frac{I}{Y} - g_y
\end{aligned}
$$

Zero Growth Steady State in Recursive Form (cont)

$$
\begin{aligned}
\frac{H\varrho}{(1-H)(1-\varrho)} &= \frac{WH/Y}{C/Y} = \frac{\alpha}{C/Y} \\
&\Rightarrow H = \frac{\alpha(1-\varrho)}{\varrho C/Y + \alpha(1-\varrho)} \\
Y &= (AH)^{\alpha} K^{1-\alpha} = (AH)^{\alpha}\left(\frac{K}{Y}\right)^{1-\alpha} (Y)^{1-\alpha} \\
&\Rightarrow Y = (AH)(K/Y)^{\frac{1-\alpha}{\alpha}} \\
G &= g_y Y = T \\
W &= \alpha \frac{Y}{H} \\
I &= \frac{I}{Y} Y\,;\; C = \frac{C}{Y} Y\,;\; K = \frac{K}{Y} Y
\end{aligned}
$$

Normalization and Calibration of RBC Model

- **Normalization**: 1 unit of raw labour + 1 unit of capital gives 1 unit of output $\Rightarrow A = 1$
- Use micro-econometric evidence on σ_c
- **Calibration**: Observe R, H and long-run shares $\frac{WH}{Y}$, $c_y \equiv \frac{C}{Y}$, $i_y \equiv \frac{I}{Y}$ and $g_y \equiv \frac{G}{Y}$ to pin down α, δ, ϱ and β
- Calibrate α to be the wage share in the wholesale sector
- δ can be calibrated using the steady state equation

$$i_y \equiv \frac{I}{Y} \quad = \quad \frac{\delta K}{Y} = \frac{\delta K}{Y} = \frac{\delta(1-\alpha)}{R-1+\delta}$$

- From the steady state equation $H = \frac{\alpha(1-\varrho)}{\varrho C/Y + \alpha(1-\varrho)}$, ϱ is obtained as:

$$\varrho = \frac{(1-H)\alpha}{(1-H)\alpha + c_y H}$$

- Finally from an observation of R we can calibrate β from $R = \frac{1}{\beta}$
- See next set of slides on *Calibration and Use of the External Steady State*.

Impulse Response Function

- This section investigates the importance of shocks to the endogenous variables of interests by analyzing the impulse response to the structural shocks in the models.
- The model first-order (linear) impulse response functions (IRFs) can be directly related to the reduced-form state space representation of the economic model.
- The RE solution of a linearized DSGE model has the general linear state space form:

$$x_t = Ax_{t-1} + B\varepsilon_t \quad (11)$$

$$y_t = Cx_t \quad (12)$$

- where x_t is the potentially unobservable state vector and y_t is the vector of the observables all measured relative to the steady state; $x_t \equiv X_t - X$, etc.
- ε_t is the vector of economic i.i.d shocks with zero mean.

Impulse Response Function

- (11) can be written as $(I - AL)x_t = B\varepsilon_t$ where L is the lag operator. Hence (12) becomes

$$y_t = C(I - AL)^{-1}B\varepsilon_t = C(1 + AL + (AL)^2 + (AL)^3 + \cdots)B$$
$$= \sum_{j=0}^{\infty} C(AL)^j B\varepsilon_t \quad (13)$$

- Hence the impulse responses from the structural shocks ε_t to Y_t are given by the following *infinite moving average (MA) representation*

$$y_t = d(L)\varepsilon_t = \sum_{j=0}^{\infty} d_j L^j \varepsilon_t = \sum_{j=0}^{\infty} d_j \varepsilon_{t-j} \quad (14)$$

  where $d_j = CA^j B$ for $j \geq 0$.
- Then with $\varepsilon_t = 0$ for $t \leq 1$, given an unanticipated shock ε_1 at $t = 1$, $y_0 = 0$, $y_1 = d_0\varepsilon_1$, $y_2 = d_1\varepsilon_1$, $y_i = d_{i-1}\varepsilon_1$, $i \geq 1$.
- $d_j \to 0$ as $j \to \infty$ provided the eigenvalues of A in are inside the unit circle (see later topic on eigenvalues).

Impulse Response Function and Dynare

- In Dynare IRFs are the future path of the endogenous variables conditional on a shock in *period 1 of one-standard-deviation*.
- The Dynare procedure runs an IRF starting from the steady state.
- One can also compute the moments and other statistics of the simulated variables variables in the model.
- Using the *stoch_simul* keyword and adding a list of variables of interest, e.g. *stoch_simul(irf=20) Y C R;* generates the IRFs.
- The argument *irf=INTEGER* tells Dynare to plot INTEGER-period IRFs. All simulation outputs from Dynare are stored in *FILENAME_results.mat*; reloading the field *oo_.irfs* from the .mat file allows us to subplot and compare the IRFs from different models.
- Following estimations (later on the Course) the initial parameter settings are replaced by the estimated posterior means.

Moment Comparisons

- The most basic RBC model has only private consumption ($G_t = 0$).
- This is the model studied in De Jong and Dave (2007), Chapter 6, Section 6.4.
- Estimating labour's share as $\alpha = 0.77$ and $\delta = 0.02$ they estimate an AR1 process for total factor productivity (not labour productivity as in our previous set-up) giving $\rho_A = 0.78$ and $\text{sd}(\epsilon_A) = 0.0067$.
- The moments for HP-filtered data we computed earlier on the Course for the full sample (computed using the code *moments_matching.m* and the corresponding ones from our model with these features (*RBC_nogov.mod*) are shown in the Table below.
- Not surprising perhaps this very simple model performs rather well in reproducing the moments for output but it does not in the case of consumption, investment and hours.

Moment Comparisons

HP Filtered Data

j	σ_j	$\frac{\sigma_j}{\sigma_Y}$	$\varphi(1)$	$\varphi(j, Y)$
Y	1.7374	1.00	0.8478	1.0
C	1.1940	0.6872	0.7717	0.7885
I	4.9960	2.8755	0.8717	0.7791
H	2.8992	1.6687	0.9676	0.5135

RBC Model

j	σ_j	$\frac{\sigma_j}{\sigma_Y}$	$\varphi(1)$	$\varphi(j, Y)$
Y	1.4432	1.00	0.7927	1.00
C	0.6409	0.4441	0.9454	0.8068
I	6.9042	4.7840	0.7441	0.9527
H	0.7047	0.4883	0.7385	0.9256

Notes: σ_Y denotes the standard deviation of $\frac{Y_t}{Y}$ etc; $\varphi(1)$ denotes first-order autocorrelation; $\varphi(j, Y)$ denotes cross-correlation with output;

The Social Planner's Problem

- The RBC model up to now is for a decentralized market economy.
- By contrast, the social (central planner's) problem to choose paths for consumption $\{C_t\}$, leisure $\{L_t\} = \{1 - H_t\}$ and capital stock $\{K_t\}$ to maximize the value function V_t given by

$$V_t = V_t(K_{t-1}) = E_t \left[\sum_{s=0}^{\infty} \beta^s U(C_{t+s}, L_{t+s}) \right] \tag{15}$$

given initial capital stock at the beginning of period t, K_{t-1}, and given *the resource constraint*

$$F(K_t, H_t) = C_t + G_t + \underbrace{\Delta K_t + \delta K_{t-1}}_{I_t} = C_t + \underbrace{K_t - (1-\delta)K_{t-1}}_{I_t} \tag{16}$$

- **Exercise**
Use the Lagrangian method of Appendix A of the Notes to show that *the allocation of* $\{C_t\}$, $\{K_t\}$, $\{L_t\}$, $\{H_t\}$ *over time is identical in the centralized and decentralized economies*

RBC Model with JR Preferences

- The RBC model up to now with a CD utility function displays a strong *wealth effect* in response to a positive technology shock.
- As a result household reduce their hours relative to the steady state and "consume" more leisure.
- Hours and output then do *not co-move*, as in the data.
- The following alternative functional form for utility found in Jaimovich and Rebello (2008) controls the wealth effect:

$$
\begin{aligned}
U_t &= \frac{(C_t - \kappa H_t^{\theta} \Xi_t)^{1-\sigma_{es}} - 1}{1-\sigma_{es}} \\
&\to \log(C_t - \kappa H_t^{\theta} \Xi_t) \text{ as } \sigma_{es} \to 1 \\
\Xi_t &= C_t^{\gamma} \Xi_{t-1}^{1-\gamma}; \quad \gamma \in [0,1]
\end{aligned}
$$

JR Preferences: Calibration of Parameters

- There are three parameters to calibrate: κ, θ and γ:
- The parameter κ can be set to target $\bar{H}$ (as we did using ϱ with the Cobb-Douglas function previously).
- The parameter θ can be set to target the elasticity of labour supply with respect to the real wage (inverse of the Frisch parameter) (See Bilbiie (2009) and Bilbiie (2011) for details.)
- This leaves γ to control for wealth effects
- The CD utility function is less flexible in that it can only target one steady state outcome $H = \bar{H}$ whereas the JR utility function can target labour supply elasticity and (as we shall see) wealth effects.

JR Preferences: Comparison with CD

- How does the JR utility function compare with the Cobb-Douglas form?. Writing the latter as

$$
\begin{aligned}
U_t &= \frac{(C_t^{(1-\varrho)}(1-H_t)^{\varrho})^{1-\sigma_c}-1}{1-\sigma_c} \\
&= \frac{(C_t(1-H_t)^{\varrho/(1-\varrho)})^{(1-\varrho)(1-\sigma_c)})-1}{1-\sigma_c} \\
&= \frac{(C_t(1-H_t)^{\theta})^{\sigma_{es}}-1}{1-\sigma_{es}} \quad \text{where} \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\theta &= \varrho/(1-\varrho) \qquad (18) \\
\sigma_{es} &= 1-(1-\varrho)(1-\sigma_c) \qquad (19)
\end{aligned}
$$

- Empirical evidence is obtained from utility function of the form (17) for the elasticity σ_{es}. From (19), we should therefore calibrate σ_c from

$$\sigma_c = \frac{\sigma_{es}-\varrho}{1-\varrho}$$

JR Preferences: Household foc

- From Appendix 1.1 of the Notes, the household first-order conditions now become:

$$\text{Euler Consumption} : 1 = R_t \mathbb{E}_t \left[\Lambda_{t,t+1} \right]$$

$$\text{Stochastic Discount Factor} : \Lambda_{t,t+1} \equiv \beta \frac{\lambda_{t+1}}{\lambda_t}$$

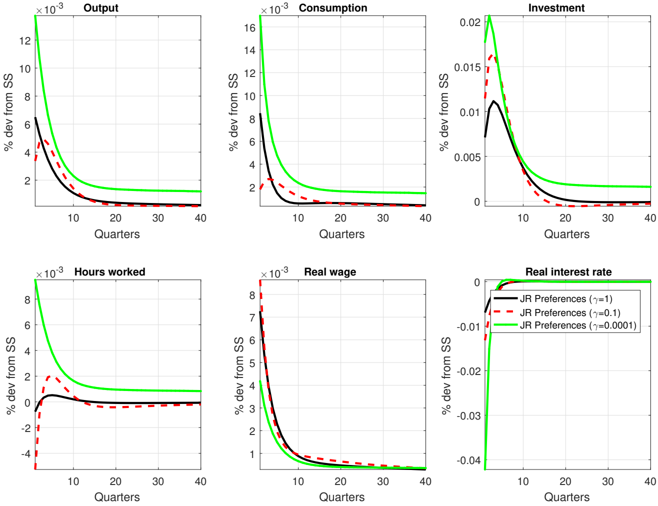
$$\text{where} : \lambda_t = U_{C,t} - \gamma \mu_t \frac{\Xi_t}{C_t}$$

$$\text{and} : \mu_t = -U_{\Xi,t} + \beta(1-\gamma)\mathbb{E}_t \frac{\mu_{t+1}\Xi_{t+1}}{\Xi_t}$$

$$\text{Labour Supply} : \frac{U_{H,t}}{\lambda_t} = -W_t$$

- Investment and capital supply foc as before
- The following irfs to a technology shock show how wealth effects are reduced by reducing γ. Note that $\gamma > 0$ is required for a bgp.

The Wealth Effect with JR Preferences



Dynare Model Files

In **folder RBC**

- The dynare model files are **RBC.mod** and **RBC_JR.mod** for CD and JR preferences respectively.
- Matlab file **graphs_irfs_compare_RBCs** will allow you to compare irfs of different RBC models.

Exercises RBC I

- Use the set-up with JR preferences and **graphs_irfs_compare_RBC** to reproduce the graphs above and discuss these irfs

Exercises RBC II

Introducing *external habit* into the utility function, the single-period utility, marginal utility of consumption and labour supply become respectively:

$$
\begin{aligned}
U_t &= U(C_t, H_t) = \frac{((C_t - \chi C_{t-1})^{(1-\varrho)}(1-H_t)^{\varrho})^{1-\sigma_c} - 1}{1-\sigma_c} \\
U_{C,t} &= (1-\varrho)(C_t - \chi C_{t-1})^{(1-\varrho)(1-\sigma_c)-1}(1-H_t)^{\varrho(1-\sigma_c)} \\
U_{H,t} &= -\varrho(C_t - \chi C_{t-1})^{(1-\varrho)(1-\sigma_c)}(1-H_t)^{\varrho(1-\sigma_c)-1}
\end{aligned}
$$

Note that in evaluating the marginal utility of consumption, $U_{C,t}$, the household takes external habit as given.

Exercises RBC II continued

- Now proceed through the following steps :
  1. Rework the analytical zero-growth steady state in recursive form as above.
  2. Use 1 to set up and run the model without an external steady state. Don't forget to rename the mod file to say **RBC_hab.mod**
  3. Use the graph plotter with appropriate changes to the file names of results to compare your RBC model with and without habit.
  4. Discuss your results.

Bilbiie, F. (2009). Nonseparable Preferences, Fiscal policy Puzzles and Inferior Goods. *Journal of Money, Credit and Banking*, **41**(2-3), 443–450.

Bilbiie, F. (2011). Nonseparable Preferences, Frisch Labor Supply and the Consumption Multiplier of Government Multiplier of Government Spending: One Solution to the Fiscal Policy Puzzle. *Journal of Money, Credit and Banking*, **43**(1), 221–251.

Jaimovich, N. and Rebello, S. (2008). News and Business Cycles in Open Economies. *Journal of Money, Credit and Banking*, **40**(8), 1699–1710.