The Science and Art of DSGE Modelling
A Foundations Course

Optimal Policy

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Four Policy Regimes

- Rules considered depend on whether the policymaker can *commit*, or she exercises *discretion* and engages in period-by-period optimization.
- With commitment the welfare-optimal policy is the solution to the *Ramsey problem*; but this is *not time-consistent* in RE models: with the mere passage of time initially optimal policy becomes sub-optimal.
- The Ramsey solution is not the same thing as the *social planner's problem* in any model with some market failure.
- In the absence of commitment the policymaker optimizes period-by-period - the *discretionary solution*. This is sub-optimal.
- Even with commitment the policymaker may be constrained to *simple rules* (e.g., Taylor-type rules)
- Rationale for simplicity: transparency, information available and ease of implementation

The Ramsey Problem: the Constraints

- Our models are all special cases of the following general setup recognized by Dynare in non-linear form

$$\mathrm{Z}_t = J(\mathrm{Z}_{t-1}, \mathrm{X}_t, \mathrm{w}_t, \epsilon_t) \quad (1)$$

$$E_t \mathrm{X}_{t+1} = K(\mathrm{Z}_t, \mathrm{X}_t, \mathrm{w}_t) \quad (2)$$

where $\mathrm{Z}_{t-1}, \mathrm{X}_t$ are $(n-m) \times 1$ and $m \times 1$ vectors of backward and forward-looking variables, respectively, ϵ_t is a $\ell \times 1$ i.i.d shock variable and w_t is an $r \times 1$ vector of instruments.

- Now define

$$\mathrm{y}_t \equiv \begin{bmatrix} \mathrm{Z}_t \\ \mathrm{X}_t \end{bmatrix}$$

- Then, as in Dynare User Guide, chapter 7, (1) and (2) can be written

$$\begin{aligned} \mathbb{E}_t[f(\mathrm{y}_t, \mathrm{y}_{t+1}, \mathrm{y}_{t-1}, \mathrm{w}_t, \epsilon_t)] &= 0 \quad (3) \\ E_{t-1}[\epsilon_t] &= 0 \\ E_{t-1}[\epsilon_t \epsilon_t'] &= \Sigma_\epsilon \end{aligned}$$

The Ramsey Problem: FOC

- The general problem is to maximize at time 0, $\Omega_0 = E_0\left[\sum_{t=0}^{\infty}\beta^t u(\mathrm{y}_t, \mathrm{y}_{t-1}, \mathrm{w}_t)\right]$ subject to (3) given initial values Z_0. To carry out this problem write the Lagrangian

$$L = E_0\left[\sum \beta^t[u(\mathrm{y}_t, \mathrm{y}_{t-1}, \mathrm{w}_t) + \lambda_t^T f(\mathrm{y}_t, \mathrm{y}_{t+1}, \mathrm{y}_{t-1}, \mathrm{w}_t, \epsilon_t]\right]$$

  where λ_t is a column vector of multipliers associated with the n constraints defining the model.

- First-order conditions are given by

$$E_0\left[\frac{\partial L}{\partial \mathrm{w}_t}\right] = E_0\left[u_3(\mathrm{y}_t, \mathrm{y}_{t-1}, \mathrm{w}_t) + \lambda_t^T f_4(\mathrm{y}_t, \mathrm{y}_{t+1}, \mathrm{y}_{t-1}, \mathrm{w}_t, \epsilon_t)\right] = 0$$

$$E_0\left[\frac{\partial L}{\partial \mathrm{y}_t}\right] = E_0[u_1(\mathrm{y}_t, \mathrm{y}_{t-1}, \mathrm{w}_t) + \beta u_2(\mathrm{y}_{t+1}, \mathrm{y}_t, \mathrm{w}_{t+1}) + \lambda_t^T f_1(\mathrm{y}_t, \mathrm{y}_{t+1}, \mathrm{y}_{t-1}, \mathrm{w}_t, \epsilon_t) + \frac{1}{\beta}\lambda_{t-1}^T f_2(\mathrm{y}_{t-1}, \mathrm{y}_t, \mathrm{y}_{t-2}, \mathrm{w}_{t-1}, \epsilon_{t-1}) + \beta\lambda_{t+1}^T f_3(\mathrm{y}_{t+1}, \mathrm{y}_{t+2}, \mathrm{y}_t, \mathrm{w}_{t+1}, \epsilon_{t+1})] = 0$$

  where the subscripts in $\{u_i, f_i\}$ refer to the partial derivatives of the ith variable in u, f.

Further Optimality Conditions

- Now partition $\lambda_t = [\lambda_{1,t}\,\lambda_{2,t}]$ so that $\lambda_{1,t}$, the co-state vector associated with the predetermined variables, is of dimension $(n-m)\times 1$ and $\lambda_{2,t}$, the co-state vector associated with the non-predetermined variables, is of dimension $m \times 1$.
- A crucial initial optimal condition is

$$\begin{aligned} \lambda_{2,0} &= 0;\ \text{(ex ante optimal)} \\ \lambda_{2,0} &= \lambda_2;\ \text{(`timeless' solution)} \end{aligned}$$

  where λ_2 is the deterministic steady state of $\lambda_{2,t}$.
- At any time $t > 0$ there then exists a gain from reneging by resetting $\lambda_{2,t} = 0$. Thus there is *an incentive to renege - the time-inconsistency problem*.
- The *timeless solution* imposes a *time-invariance* (not time-consistency!) on the solution.

Discretionary Policy

- To evaluate the time-consistent (discretionary) policy we write the expected loss Ω_t at time t in Bellman form as[1]

$$\Omega_t = \mathbb{E}_t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} u(\mathrm{y}_t, \mathrm{y}_{t-1}, \mathrm{w}_t) \right] = u(\mathrm{y}_t, \mathrm{y}_{t-1}, \mathrm{w}_t) + \beta \mathbb{E}_t \left[\Omega_{t+1} \right] \quad (4)$$

- The dynamic programming solution then seeks a stationary *Markov Perfect* solution of the form $\mathrm{w}_t = F(\mathrm{Z}_t)$, and $\mathrm{X}_t = G(\mathrm{Z}_t)$. Ω_t is maximized at time t, subject to the model constraints, in the knowledge that a similar procedure will be used to minimize Ω_{t+1} at time $t+1$.[2]

[1] This applies only to the zero-growth steady state.

[2] See [Currie and Levine(1993)] and [Söderlind(1999)] for a LQ treatment of this problem.

Welfare-Optimal Simple Rules

- Optimal policy in the form of the Ramsey solution can be expressed as $\mathrm{w}_t = f(\mathrm{Z}_t, \lambda_{2,\mathrm{t}})$. This poses problems for the implementability of policy in terms of complexity and the observability of elements of Z_t (such as the technology process A_t, but more importantly $\lambda_{2,\mathrm{t}}$).
- The macroeconomic policy literature therefore focuses on **simple rules**, using the Ramsey solution as a benchmark. All our rules take the log-linear form

$$\log \mathrm{w}_t = D \log \mathrm{y}_t$$

  where we define $\log \mathrm{w}_t \equiv [\log w_{1,t} \log w_{2,t}, \cdots, \log w_{r,t}]^T$ over r instruments, and similarly for $\log \mathrm{y}_t$, and the matrix D selects a subset of y_t from which to feedback.
- This is quite general in that y_t can be enlarged to include lagged and forward-looking variables.

Computing Welfare-Optimal Simple Rules in Dynare

- The optimized simple rules then defines the inter-temporal welfare loss at time t in Bellman form (4), sets steady-state values for instruments w_t, denoted by w, computes a second-order solution for a particular setting of w and, given initial values Z_0, solves the **maximization problem** at $t = 0$,

$$\max_{\mathrm{w}, D} \Omega_0(\mathrm{Z}_0, \mathrm{w}, D)$$

- In a **purely stochastic problem** we put $\mathrm{Z}_0 = \mathrm{Z}$, the steady state of Z_t, maximizing the conditional welfare at the steady state.
- In a **purely deterministic problem** there is no exogenous uncertainty and the optimization problem is driven by the need to return from Z_0 to its steady state, Z.
- Of the three policy rules discussed up to now, only the Ramsey solution is available in Dynare for a non-linear model and a general objective function. However **linear-quadratic problems** are available for all three forms of policy and are now examined.

The Linear Quadratic Special Case

- Consider our first NK Model linearized around a zero-growth, zero-inflation steady state. A standard ad hoc quadratic loss function in deviation form is given by

$$\begin{aligned}\Omega_0 &= (1-\beta)E_0\left[\sum_{t=0}^{\infty}\beta^t(16\pi_t^2 + y_t^2 + r_{n,t}^2)\right] \\ &\simeq 16\text{var}(\pi_t) + \text{var}(y_t) + \text{var}(r_{n,t}) \text{ as } \beta \to 1 \qquad (5)\end{aligned}$$

  where the variances are conditional variances at the steady state.

- Dynare 4.5.5 computes the Ramsey policy (optimal commitment), discretionary policy and optimized simple rules for the LQ set-up as in this section.

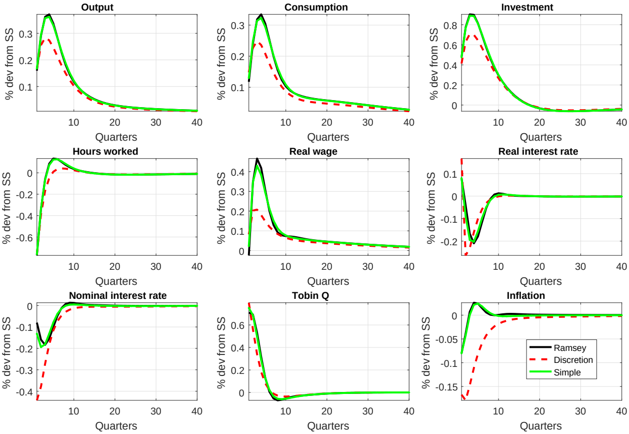
The LQ Case: Results

Rule	Ω_0	ρ	α_π	α_y	c_e	sd(R_n)
Ramsey	1.310	n.a	n.a.	n.a.	0	0.41
Discretion	3.412	n.a	n.a.	n.a.	0.023	0.50
Simple	1.557	0.993	0.476	0.003	0.003	0.31

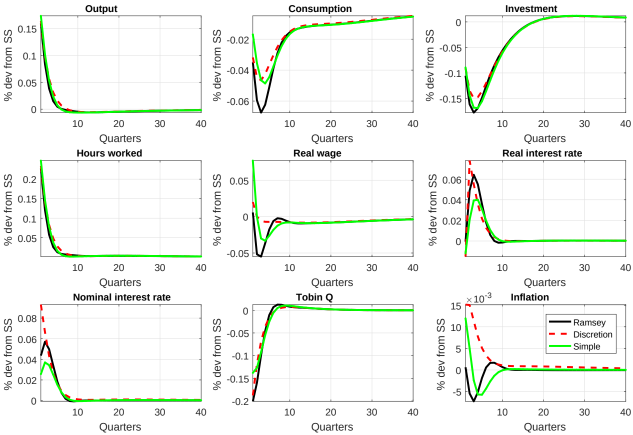
Table: LQ Framework: Optimal Policy, Discretionary Policy and Optimized Rules

- We report the expected inter-temporal loss as given by (5).
- The loss X relative to the Ramsey problem is expressed in terms of a consumption equivalent percentage increase ($c_e \equiv \frac{\Delta C}{C} \times 10^2$)
- From Notes, $c_e = \frac{X \times 10^{-2}}{CU_C}$
- For the steady state of this model, $CU_C = 0.8983$. It follow that a welfare loss difference of $X = 1$ gives a consumption equivalent percentage difference of 0.0113%.
- Steady-state variances of the nominal interest rate highlight zero lower bound (ZLB) problems.

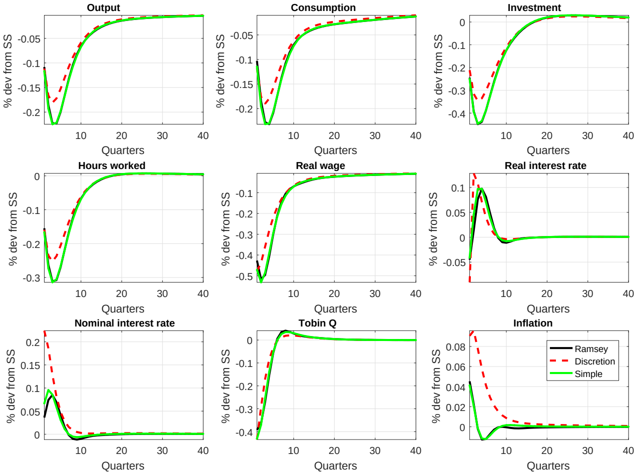
Impulse Responses to a Technology Shock



Impulse Responses to a Gov Spending Shock



Impulse Responses to a Mark-up Shock



Dynare, Matlab Codes and Exercises

1. **Dynare Code**:Dynare mod file **NKlinear_policy.mod** provides options for the Ramsey, Discretion and Implementable Simple Rules LQ exercises with a preprocessor command to select your choice. Matlab file **graphs_irfs_compare_NK_optimal_policy** compares irfs for these three policy regimes.
2. Rework the LQ analysis in Table above for a 'conservative banker' with welfare loss given by

$$\begin{aligned}\Omega_0 &= (1-\beta)\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t(32\pi_t^2 + y_t^2 + r_{n,t}^2)\right] \\ &\simeq 32\text{var}(\pi_t) + \text{var}(y_t) + \text{var}(r_{n,t}) \text{ as } \beta \to 1\end{aligned}$$

3. Rework Table 1 using the conventional Taylor rule. What do you notice about the results?

Currie, D. and Levine, P. (1993).
Rules, Reputation and Macroeconomic Policy Coordination.
Cambridge University Press.

Söderlind, P. (1999).
Solution and Estimation of RE Macromodels with Optimal Policy.
European Economic Review, **43**, 813–823.